Lecture 1 3/3/18

Modular Forms

Used to -

* Study diophantine equations
* Quadratic Forms
* Models for mechanical systems which display "chaos"

Quadratic Form - $Q(x_1, \ldots, x_d) = \sum_{i \le j} a_{ij} x_i x_j$ (e.g. $Q(x_1, \ldots, x_d) = x_1^2 + \ldots + x_d^2$)
$a_{ij} \in \mathbb{Z}$

Which integers are represented by Q, that is, $n \overset{?}{=} Q(x_1, \ldots, x_d)$ for $\underline{x} = (x_1, \ldots, x_d) \in \mathbb{Z}^d$.

$n = x^2 + y^2$?

$n = x^2 + y^2 + z^2$? $\qquad r_Q(n) = \# \{ x \in \mathbb{Z}^d \mid Q(x) = n \}$

$n = x^2 + y^2 + z^2 + w^2$?

Thm: (Fermat)

p prime, $p = x^2 + y^2 \Leftrightarrow p = 2, p \equiv 1 \mod 4$

$n \geq 0$ integer, $n = x^2 + y^2 \Leftrightarrow$ factor $n = 2^a \cdot \prod_{p_i \equiv 1 (4)} p_i^{a_i} \cdot \prod_{q_j \equiv 3 (4)} q_j^{b_j}$, b_j - even

(Taught in basic NT course)

Moreover, when $Q = x^2 + y^2$, $r_Q(p) = \begin{cases} 4, & p = 2 \\ 0, & p \equiv 3 (4) \\ 8, & p \equiv 1 (4) \end{cases}$

$2 = (\pm 1)^2 + (\pm 1)^2$

$5 = (\pm 2)^2 + (\pm 1)^2$
$= (\pm 1)^2 + (\pm 2)^2$

To study $r_Q(n)$, use a θ-function $\Theta_Q(\tau) = \sum_{\vec{m} \in \mathbb{Z}^d} e^{2\pi i \tau Q(\vec{m})} =$

$= \sum_n r_Q(n) e^{2\pi i n \tau}$

We have -

1) $\theta(\tau + 1) = \theta(\tau)$ (Trivial)

Will have extra symmetries.

Example: $Q(x)=x^2$, $\Theta_1(\tau)=\sum_{n=-\infty}^{\infty} e^{2\pi i n^2\cdot\tau}$, $Im(\tau)>0, \tau=x+iy$

$\tau\mapsto -1/\tau$ gives:

$$\Theta_1(-1/\tau) = \sum_{n=-\infty}^{\infty} e^{\frac{1}{|\tau|^2}2\pi i n^2 x}\, e^{-2\pi n^2\cdot y/|\tau|}$$

is analytic in $\mathbb{H}=\{\tau=x+iy \mid y>0\}$ $\longrightarrow$ space of lattices

$$\Theta(-1/\tau)=\sqrt{\tau}\,\Theta(\tau)$$

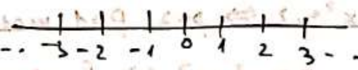
Example: $\Theta_2(\tau)=\sum_{(x,y)\in\mathbb{Z}^2} e^{2\pi i\tau(x^2+y^2)} = \Theta_1(\tau)^2$

In some cases, the constraints of symmetry will give us uniqueness; and so if by other methods we can construct a function with these properties, we get it, in a way which is easier to calculate its coefficients.

The space of Lattices

Def: A Lattice in $\mathbb{R}^d$ is a discrete subgroup $L\subseteq\mathbb{R}^d$ which spans $\mathbb{R}^d$.

Example: 1) $L=\mathbb{Z}\subseteq\mathbb{R}^1$



2) $L=\mathbb{Z}^2\subseteq\mathbb{R}^2$

Def: A discrete set $S\subseteq\mathbb{R}^d$ is a set without accumulation points, or a set in which every point is isolated:

$$\forall s\in S.\ \exists \varepsilon(s)>0.\ |s'-s|\geq\varepsilon(s),\ \forall s'\in S$$

Comments: Because L is a subgroup we can take $\varepsilon(l)=\varepsilon(0)>0$ independent of l.

$\varepsilon(0) = \inf\{|0-l'| \mid 0\neq l'\in L\}$ = length of a shortest non zero vector.

Note that $\{(n,0) \mid n\in\mathbb{Z}\} \subseteq \mathbb{R}^2$ is a discrete subgroup of $\mathbb{R}^3$ but doesn't span $\mathbb{R}^2$, so it's not a lattice.

Thm: Every lattice has a basis $L\subseteq\mathbb{R}^d$, So, $\exists w_1,\dots,w_d\in\mathbb{R}^d$ linearly independent and span $\mathbb{R}^d$, and $L=\mathbb{Z}w_1+\dots+\mathbb{Z}w_d$.

(Why, if we have this result, to take the abstract definition? Just because the abstract way is the way we encounter lattices "in nature" - say an elliptic function is a function with 2 linearly ind. periods; then the space of periods turns out to be a lattice)

Comment: Every subset of L has a shortest, non zero, vector. o/w since $0\in L$, we have that 0 is an accumulation point

Proof of the thm: (Only for $d\leq 2$ - higher d is a exercise)

$d=1$ - Want $L\subseteq\mathbb{R}^1 \Rightarrow L=\mathbb{Z}w_1$, $w_1\neq 0$

Take $0\neq w_1$ a shortest vector. If need replace $w_1\to -w_1$ to have $w_1>0$. Want $L=\mathbb{Z}w_1$. O/w, $\mathbb{Z}w_1\subsetneq L$. Assume $\exists l\in L-\mathbb{Z}w_1$.

So, $\exists n\in\mathbb{Z}$ s.t $nw_1<l<(n+1)w_1$

$\Rightarrow 0<l-nw_1<w_1$, $l-nw_1\in L$

contradicting the minimallity of w_1.

$d=2$ Will show $L=\mathbb{Z}w_1+\mathbb{Z}w_2$, where w_1 is a shortest non zero vector, w_2 is a shortest vector linearly ind. of w_1.

Claim: This choice of w_1,w_2 gives $L=\mathbb{Z}w_1+\mathbb{Z}w_2$

clearly, "$\supseteq$" holds. look at $L\cap\mathbb{R}w_1\subseteq\mathbb{R}w_1$, a one-dim Lattice. (Obviously a subgroup and discrete)

By dim 1 case, $L\cap\mathbb{R}w_1=\mathbb{Z}w_1$. Want $L=\mathbb{Z}w_1+\mathbb{Z}w_2$.

Assume $l\in L$. Write $l=x_1w_1+x_2w_2$, $x_i\in\mathbb{R}$.

Firstly note $n_2 x_2\in\mathbb{Z}\Rightarrow x_2\in\mathbb{Z}$, since then,

$l-n_2w_2\in L\cap\mathbb{R}w_1$, so $l-n_2w_2\in\mathbb{Z}w_1$, and $x_1\in\mathbb{Z}$ as well.

Likewise, $L \cap \mathbb{R}w_2 = \mathbb{Z}w_2$ so we'll get $x_1 \in \mathbb{Z} \Rightarrow x_2 \in \mathbb{Z}$. So we can assume $x_1, x_2 \notin \mathbb{Z}$.

Choose integers $n_1, n_2 \in \mathbb{Z}$ s.t. $0 < |x_1 - n_1| \leq 1/2$
$0 < |x_2 - n_2| \leq 1/2$

Look at; $l' = l - (n_1 w_1 + n_2 w_2) \in L$
$= (x_1 - n_1) w_1 + (x_2 - n_2) w_2$

First attempt: I want to show $|l'| < |w_2|$ and is lin. ind. of $w_1 \Rightarrow$ contradiction.

$$|l'| \underset{(*)}{\leq} |x_1 - n_1||w_1| + |x_2 - n_2||w_2| \leq \frac{1}{2}|w_1| + \frac{1}{2}|w_2| \underset{|w_1| \leq |w_2|}{\leq}$$

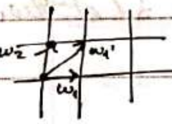
$\leq |w_2|$ - Fails...

We can fix - in $(*)$ there's a strict inequality since (as $x_1 - n_1, x_2 - n_2 \neq 0,0$) we have lin. ind.

General argument - the same idea. □

Changes of Basis

Say $L = \mathbb{Z}w_1 + \mathbb{Z}w_2 = \mathbb{Z}w_1' + \mathbb{Z}w_2'$



we have - $w_1' = a w_1 + b w_2$
$w_2' = c w_1 + d w_2$ $\quad a, b, c, d \in \mathbb{Z}$
$w_1 = a' w_1' + b' w_2'$ $\quad a', b', c', d' \in \mathbb{Z}$
$w_2 = c' w_1' + d' w_2'$

Here, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ are integral with integral matrices

So are in $GL_2(\mathbb{Z})$.

Lemma: Let $A \in Mat_2(\mathbb{Z})$. Then $A^{-1} \in Mat_2(\mathbb{Z}) \Leftrightarrow \det A$ is an invertible integer $\Leftrightarrow \det A = \pm 1$.

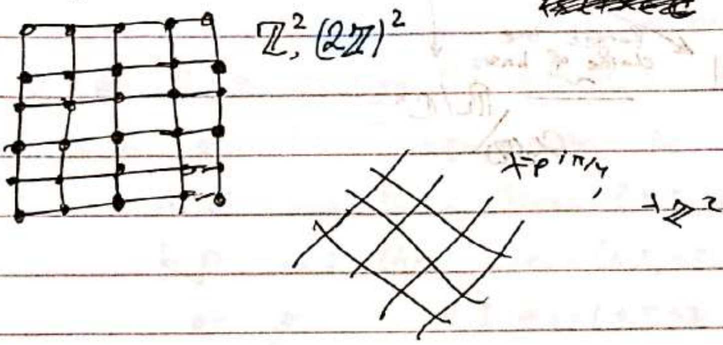
Proof: $(\Rightarrow)$ Write $A \cdot B = I$, $A, B \in Mat_2(\mathbb{Z})$, then $\det(A) \cdot \det(B) = 1$, and $\det(A), \det(B) \in \mathbb{Z}$ since A, B has entries in $\mathbb{Z}$

$(\Leftarrow)$ We have (Kummer's thm) -

$$M^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$ □

The space of 2-dim lattices

Homothety Classes: If $\lambda \in \mathbb{C}^\times$, $L \subseteq \mathbb{R}^2 = \mathbb{C}$ a lattice, we can get another lattice, $\lambda L \subseteq \mathbb{C}$.



Def: $L, \lambda L$ are Homothetic.

Let $\mathcal{R} = \{$all latices in $\mathbb{C}\}$

$\mathcal{R}/\mathbb{C}^\times = \{$Homothety classes$\}$

Goal: Geometric picture of $\mathcal{R}/\mathbb{C}^\times$ – 

Let $\mathcal{M} = \{(\omega_1,\omega_2)\in\mathbb{C}^2 \mid Im(\omega_1/\omega_2) > 0\}$. → non zero since are linearly ind.

We have a map, surjective,

$$\mathcal{M} \twoheadrightarrow \mathcal{R}$$

$$(\omega_1,\omega_2)\mapsto L(\omega)=\mathbb{Z}\omega_1+\mathbb{Z}\omega_2 \qquad (\tau = x+iy)$$

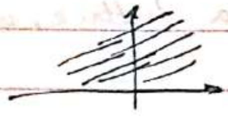
(Surjective- $L=\mathbb{Z}\omega+\mathbb{Z}\omega' = \mathbb{Z}\omega'+\mathbb{Z}\omega$, $\tau=\omega/\omega' \notin\mathbb{R}$; $Im(\omega'/\omega) = Im(1/\tau)$,
So if is <0, we can switch- $\tau\bar{\tau} = |\tau|^2>0$, $\frac{1}{\tau} = \frac{\bar{\tau}}{|\tau|^2}$, $Im(\frac{1}{\tau}) = -\frac{y}{|\tau|^2}$)

Note that by switching $(\omega_1,\omega_2)\rightsquigarrow(\lambda\omega_1,\lambda\omega_2)$ we get
$Im(\omega_1/\omega_2) = Im(\lambda\omega_1/\lambda\omega_2)$

So we have a map,

$$\mathcal{M}\longrightarrow\mathbb{H} = \{\tau = x+iy,\ y>0\}$$

$$(\omega_1,\omega_2)\mapsto \tau=\omega_1/\omega_2$$



$$\begin{tikzcd}
\mathcal{M} \arrow[r] \arrow[d] & \mathcal{R} \arrow[d] \\
\mathbb{H} \arrow[d] & \mathcal{R}/\mathbb{C}^\times \arrow[d] \arrow[dl, "\sim" ', "\text{forget the choice of basis}"] \\
SL_2(\mathbb{Z})\backslash\mathbb{H} \arrow[r, dash, "\sim"] & \mathcal{R}/\mathbb{C}^\times \big/ GL_2(\mathbb{Z})
\end{tikzcd}$$

The claim is-

(1) Onto

(2) Homothety invariance

(3) Determinant $= \pm 1$.

(3)- $w_1' = a w_1 + b w_2$

$w_2' = c w_1 + d w_2$

$\mathrm{Im}(w_1'/w_2') = \dfrac{ad-bc}{|c\frac{w_1}{w_2}+d|^2}\cdot \mathrm{Im}\left(\frac{w_1}{w_2}\right)$ (So need $ad-bc>0$ for $\mathrm{Im}(w_1'/w_2')>0$)

Conclusion:

Homothety classes of lattices $\longleftrightarrow$ $SL_2(\mathbb{Z})\backslash\mathbb{H}$

$\parallel$

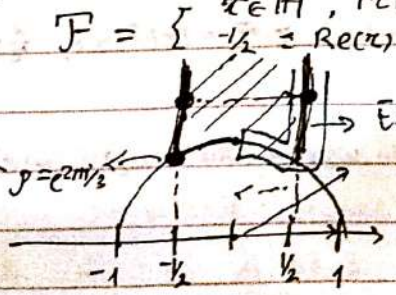
$\mathcal{R}/\mathbb{C}^\times$

$\tau \sim \dfrac{a\tau+b}{c\tau+d}$, $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\mathbb{Z})$

Define-

$$\mathcal{F} = \left\{ \begin{array}{l} \tau \in \mathbb{H},\ |\tau| \geq 1 \\ -1/2 \leq \mathrm{Re}(\tau) < 1/2,\ \text{if } |\tau| = 1 \text{ then } \mathrm{Re}(\tau) \leq 0 \end{array} \right\}$$

$\frac{-1+i\sqrt{3}}{2} = \rho = e^{2\pi i/3}$



Excluding these from the boundary

Thm:

a) $\forall \tau \in \mathbb{H}$, $\exists g \in SL_2(\mathbb{Z}) = \Gamma$ s.t $g(\tau) \in \mathcal{F}$

b) If $\tau \neq \tau' \in \bar{\mathcal{F}}$ are equivalent by Γ, then they lie on the boundary, and either $\mathrm{Re}(\tau) = -1/2$, $\tau' = \tau + 1 = \frac{1\cdot\tau+1}{0\cdot\tau+1}$
or $|\tau| = 1$, and $\tau' = -1/\tau$, $= \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}(\tau)$

c) Let $I(\tau) = \{g \in \Gamma \mid g\tau = \tau\}$. Then, $I(\tau) = \{\pm 1\}$ unles τ is Γ-equivalent to
  a) i, and then $I(\tau) = \{\pm I, \pm S\}$, $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
  b) ρ, and then $I(\tau) = \{\pm I, \pm ST, \pm (ST)^2\}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
  or $\bar{\rho}$, $I(\tau) = \{\pm I, \pm TS, \pm (TS)^2\}$

d) $SL_2(\mathbb{Z}) = \langle S, T \rangle$

Proof: (only of (a),(d))

(a) Take $\tau \in \mathbb{H}$, let $\Gamma' = \langle S, T\rangle \subseteq SL_2(\mathbb{Z})$. Want $\exists g \in \Gamma'$, s.t $g\tau \in \mathcal{F}$. Look at the set $\{c\tau + d \mid \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma'\} \subseteq \mathbb{Z}\tau + \mathbb{Z}\cdot 1 = L$

It has a shortest vector, $c_0\tau + d_0$, $g = \begin{pmatrix} * & * \\ c_0 & d_0 \end{pmatrix} \in \Gamma'$.

$\Leftrightarrow \underbrace{\mathrm{Im}(g_0\tau)}_{= \frac{\mathrm{Im}(\tau)}{|c_0\tau + d_0|^2}}$ is maximal among $\{\underbrace{\mathrm{Im}(g\tau)}_{= \frac{\mathrm{Im}(\tau)}{|c\tau+d|}} \mid g \in \Gamma'\}$

Claim: $|g_0\tau| \geq 1$. $\quad (S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})$

O/w look at $\underbrace{(S\cdot g_0)}_{\Gamma'\ni}\cdot \tau = -\frac{1}{g_0\tau}$. $\mathrm{Im}(Sg_0\tau) = \frac{\mathrm{Im}(g_0\tau)}{|g_0\tau|^2} > \mathrm{Im}(g_0\tau)$ - contradiction.

Next, Find $n \in \mathbb{Z}$ s.t $-1/2 \leq \mathrm{Re}(g_0\tau) - n < 1/2$. So replace $g_0\tau$ by $T^{-n}g_0\tau$. $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}: \tau \mapsto \tau + 1$

with: 1) $-\frac{1}{2} \leq \mathrm{Re}(\tau') < \frac{1}{2}$

2) $\mathrm{Im}(\tau') = \mathrm{Im}(g_0\tau) \Rightarrow$ is still maximal! and so, τ' is outside of the unit circle.

Thus $\tau' \in \Gamma'\tau$, lies in $\bar{\mathcal{F}}$.

(d) $\Gamma' = \langle S, T\rangle = \Gamma = SL_2(\mathbb{Z})$.

$S = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$

Example: $A = \begin{pmatrix} 4 & 9 \\ 3 & 7 \end{pmatrix} \in SL_2(\mathbb{Z})$

Note. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow S \cdot A = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ -a & -b \end{pmatrix}$.

$$T^n A = \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+nc & b+nd \\ c & d \end{pmatrix}$$

$$A = \begin{pmatrix} 4 & 9 \\ 3 & 7 \end{pmatrix} \mapsto T^{-1}A = \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix} \to ST^{-1}A = \begin{pmatrix} 3 & 7 \\ -1 & -2 \end{pmatrix} \mapsto$$

$$T^3 S T^{-1} A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$$

$$S T^3 S T^{-1} A = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix} = -T^2 = S^2 T^2$$

$$\Rightarrow A \in \langle S, T \rangle$$

This is the euclidean algorithm - clearly it works, and $SL_2(\mathbb{Z}) = \langle S, T \rangle$.

Next time - using this fundamental domain discuss quadratic forms, moving the form to the fundamental domain.