

Lecture 8:

12/12/2017

Reminder:

GUE random matrices

$$N \times N \begin{pmatrix} \underbrace{\begin{pmatrix} \xi_{11} & \xi_{12} + i\eta_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix}}_N \\ \text{Herm} \\ \underbrace{\xi_{nn}}_N \end{pmatrix} \quad \xi, \eta \text{ IID } N(0,1), \lambda_1^N \leq \dots \leq \lambda_N^N$$

$\mu_1, \dots, \mu_N$ for the unordered eigenvalues - applying a uniform permutation to $(\lambda_1^N, \dots, \lambda_N^N)$.

Lemma:

joint density of $\mu_1, \dots, \mu_p$ is:

$$\text{density: } \frac{(N-p)!}{N!} \det_{i,j=1}^p K^{(N)}(x_i, x_j)$$

$$\text{kernel: } K^{(N)}(x, y) = \sum_{k=0}^{N-1} \psi_k(x) \psi_k(y)$$

Hermite Poly:

$$h_n(x) := (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}$$

Oscillator wave functions:

$$\psi_n(x) := \frac{e^{-\frac{x^2}{4}} h_n(x)}{(\sqrt{2\pi} n!)^{1/2}}$$

Lemma: (gap prob.)

For any meas. $A \subseteq \mathbb{R}$:

$$p\left(\bigcap_{i=1}^N (\sqrt{N} \lambda_i \notin A)\right) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{A/\sqrt{N}} \int_{A/\sqrt{N}} \det_{i,j=1}^k (K^{(N)}(x_i, x_j)) \prod_{i=1}^k dx_i$$

how to think of A :

$$[-2N, 2N] \quad \underbrace{\quad \quad \quad}_{\widetilde{e}} \quad \quad \quad [-2N, 2N]$$

Fredholm determinant.

Thm. (Gaudin-Mehta):

For A compact:

$$\lim_{N \rightarrow \infty} p\left(\bigcap_{i=1}^N (\sqrt{N} \lambda_i \in A)\right) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_A \dots \int_A \det_{i,j=1}^k (S(x_i, x_j)) \prod_{i=1}^k dx_i$$

$$\text{Sine kernel: } S(x, y) := \frac{1}{\pi} \frac{\sin(x-y)}{x-y}$$

Main step in proof:

$$\text{Define } S^{(n)}(x, y) := \frac{1}{\sqrt{n}} K^{(n)}\left(\frac{x}{\sqrt{n}}, \frac{y}{\sqrt{n}}\right)$$

Lemma:

$\lim_{n \rightarrow \infty} S^{(n)}(x, y) = S(x, y)$ uniformly for x, y in any given compact set.

Missing still:

- Prove lemma in main step.
- Prove continuity of Fredholm determinants.

Fredholm determinants:

Let X be a locally compact Polish space (e.g. $X = \mathbb{R}^d$) & ν a complex measure on X (with Borel σ -algebra), e.g., Lebesgue measure on an interval in $\mathbb{R}$.

$$\text{Assume: } \|\nu\|_1 := \int_X |\nu| dx < \infty$$

Def:

A kernel is a Borel-measurable function:

$$K: X \times X \rightarrow \mathbb{C} \text{ s.t. } \|K\| := \sup_{x, y \in X} |K(x, y)| < \infty$$

Aside: Hadamard inequality?

For any $k \times k$ matrix A , $|\det(A)| \leq \prod_{i=1}^k \|A_i\|_2$ $\left\| \begin{matrix} \text{rows of } A \end{matrix} \right\|_2$

$$\Rightarrow |\det A| \leq \max_{i,j} |A_{ij}|^k \cdot k^{k/2}$$

Proof:

$$\det A = \prod_{i=1}^k \|A_i\|_2 \det B \text{ where } B_i = A_i / \|A_i\|_2$$

Let $m := B \cdot B^*$ so that diagonal entries of m are 1 & m is positive semi-definite.

$$\det(m) = \prod_{i=1}^k \lambda_i(m) \leq \left(\frac{1}{k} \sum_{i=1}^k \lambda_i(m) \right)^k = \left(\frac{1}{k} (\text{tr } m) \right)^k = 1$$

eigenvalues arithmetic-geometric mean.

$$\Rightarrow \det(A) = \prod_{i=1}^k \|A_i\|_2 \cdot \underbrace{\sqrt{\det m}}_{\leq 1}$$

Lemma:

For any two kernels $K, Z: \forall x_1, \dots, x_k \in X$.

$$\left| \det_{i,j=1}^k K(x_i, x_j) - \det_{i,j=1}^k Z(x_i, x_j) \right| \leq K^{1+k/2} \max(\|K\|, \|Z\|)^{k-1} \|K-Z\|$$

As a special case of the proof:

$$\left| \det_{i,j=1}^k K(x_i, x_j) \right| \leq K^{k/2} \|K\|^k$$

Hadamard inequality

Proof:

$$\begin{aligned} & \det \begin{pmatrix} K(x_1, x_1) & K(x_1, x_2) & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} - \det \begin{pmatrix} Z(x_1, x_1) & Z(x_1, x_2) & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} = \\ & = \sum_{i=1}^k \det \begin{pmatrix} Z(x_1, x_1) & Z(x_1, x_2) & \dots \\ \vdots & \vdots & \ddots \\ Z(x_i, x_1) & Z(x_i, x_2) & \dots \\ \vdots & \vdots & \ddots \\ (K-Z)(x_i, x_1) & \dots & \dots \\ K(x_{i+1}, x_1) & \dots & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \leq K \max(\|K\|, \|Z\|)^{k-1} \|K-Z\| K^{k/2} \end{aligned}$$

Def:

The Fredholm determinant of K, D :

$$\Delta(K) = \Delta(K, D) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \Delta_k(K, D)$$

$$\text{where } \Delta_k(K, D) = \int_X \dots \int_X \det_{i,j=1}^k K(x_i, x_j) \prod_{i=1}^k dD(x_i)$$

$$\text{where } \Delta_0(K, D) := 1$$

Prev. lemma shows it is well defined. (absolute convergence of sum)

Conclusion:

For two kernels K, Z :

$$|\Delta(K) - \Delta(Z)| \leq \sum_{k=1}^{\infty} \frac{K^{1+k/2} \|D\|^k \max(\|K\|, \|Z\|)^{k-1} \|K-Z\|}{k!}$$

$$\Delta(K) - \Delta(Z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (\Delta_k(K) - \Delta_k(Z))$$

Remark: If K has rank n , $K(x, y) = \sum_{i=1}^n f_i(x) g_i(y)$
 $\Delta(K) = \det_{i,j=1}^n (I_{K \times K} - \int f_i(x) g_j(x) dV(x))$
 $\Delta(K) = \det(I - K)$

Additional Prop. of Hermite Poly.:

Reminder: $h_0(x) = 1$, $h_{n+1}(x) = x h_n(x) - h'_n(x)$
 $\int h_k(x) h_l(x) e^{-x^2/2} \frac{1}{\sqrt{2\pi}} dx = k! \delta_{k,l}$

Lemma:

- 1) $x h_n(x) = h_{n+1}(x) + n h_{n-1}(x) \quad \forall n \geq 1$ (Three-term recurrence)
- 2) $h'_n(x) = n h_{n-1}(x)$
- 3) $\sum_{k=0}^{n-1} \frac{h_k(x) h_k(y)}{k!} = \frac{h_n(x) h_{n-1}(y) - h_{n-1}(x) h_n(y)}{(n-1)! (x-y)}$ for all $x \neq y$

(Christoffel - Darboux formula)

Proof:

(2) follows from (1) & the recursion.

For (1), develop $x h_n(x)$ in the basis $h_0, \dots, h_{n+1}$

$$x h_n(x) = \sum_{k=0}^{n+1} \frac{1}{k!} \int y h_n(y) h_k(y) \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy \cdot h_k(x)$$

As $\text{span}(h_0, \dots, h_n) = \text{poly. of deg } n$, it follows that k th term is 0 when $n > k+1$.

Remaining terms have $k = n-1, k=n, k=n+1$. When $k=n$, $y h_n(y)$ & $h_n(y)$ are even & odd functions leading to zero integral.

$$k = n+1: \frac{1}{(n+1)!} \int y h_n(y) h_{n+1}(y) \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy = 1$$

monic poly. of deg $n+1 =$
 $= \sum_{m=0}^{n+1} a_m h_m$ with $a_{n+1} = 1$

$$k = n-1: \frac{1}{(n-1)!} \int h_n(y) \cdot y \cdot h_{n-1}(y) \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy = \frac{n!}{(n-1)!} = n$$

monic poly of degree n .

Property (3):

By induction on n :

For $n=1$: $1 = \frac{x-y}{x-y}$ immediate.

For larger n : $2HS(x-y) = (x-y) \sum_{k=0}^{n-1} \frac{h_k(x)h_k(y)}{k!} =$
 $= \frac{h_{n-1}(x)h_{n-2}(y) - h_{n-2}(x)h_{n-1}(y)}{(n-2)!} + (x-y) \cdot \frac{h_{n-1}(x)h_{n-1}(y)}{(n-1)!}$
 apply three term recurrence

As an immediate corollary:

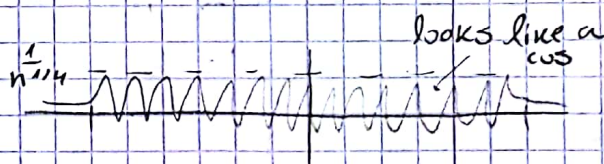
- 1) $\int \Psi_n(x) \Psi_n(x) dx = \delta_{n,l}$
- 2) (3-term recurrence): $x \Psi_n(x) = \sqrt{n+1} \Psi_{n+1}(x) + \sqrt{n} \Psi_{n-1}(x)$
- 3) Christoffel-Darboux: $\sum_{k=0}^{n-1} \Psi_k(x) \Psi_k(y) = \sqrt{n} \frac{(\Psi_n(x) \Psi_{n-1}(y) - \Psi_{n-1}(x) \Psi_n(y))}{x-y}$
- 4) $\Psi_n'(x) = -\frac{x}{2} \Psi_n(x) + \sqrt{n} \Psi_{n-1}(x)$

Convenient def:

$$\bar{\Psi}_r(x) := n^{1/4} \Psi_r\left(\frac{x}{\sqrt{n}}\right)$$

where it is understood that $|n-r|$ is a fixed number.

Lemma:



For fixed $|n-r|$,

$$\lim_{n \rightarrow \infty} \left| \bar{\Psi}_r(t) - \frac{1}{\sqrt{\pi}} \cos\left(t - \frac{\pi r}{2}\right) \right| = 0$$

uniformly for t in a given bounded interval.

Proof of main Lemma:

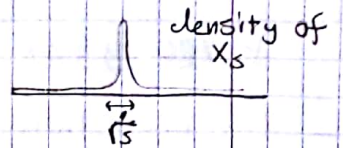
$$S^n(x,y) = \frac{1}{\sqrt{n}} K^{(n)}\left(\frac{x}{\sqrt{n}}, \frac{y}{\sqrt{n}}\right) = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \Psi_k\left(\frac{x}{\sqrt{n}}\right) \Psi_k\left(\frac{y}{\sqrt{n}}\right) \stackrel{CD}{=} 0$$

$$= \sqrt{n} \frac{(\bar{\Psi}_n(\frac{x}{\sqrt{n}}) \bar{\Psi}_{n-1}(\frac{y}{\sqrt{n}}) - \bar{\Psi}_{n-1}(\frac{x}{\sqrt{n}}) \bar{\Psi}_n(\frac{y}{\sqrt{n}}))}{x-y}$$

The previous lemma shows that for any $x \neq y$ this converges to $S(x,y)$.

To get rid of the $x-y$ factor, can express this using derivative
 use $\frac{f(x)g(y) - f(y)g(x)}{x-y} = g(y) \int_0^1 f'(t) dx + (1-t)y dt - f(y) \int_0^1 g'(t) x + (1-t)y dt$
 this allows the uniform conv. using the formula for Ψ_n' .

Laplace method (Saddle Point method):



Idea:

Suppose $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ is ^{continuous} odd. Let $X_s \sim N(0, \frac{1}{s})$
 $E[\alpha(X_s)] \xrightarrow{s \rightarrow \infty} \alpha(0)$ e.g., by Chebyshev

$$\int_{-\infty}^{\infty} \alpha(x) \frac{\sqrt{s}}{\sqrt{2\pi}} e^{-\frac{sx^2}{2}} dx$$
$$\int_{-\infty}^{\infty} \alpha(x) e^{-\frac{sx^2}{2}} dx \sim \frac{\alpha(0)}{\sqrt{s}} \sqrt{2\pi}$$

More generally, let $\delta(x)$ be smooth & have a maximum at a .

$$\text{As } s \rightarrow \infty: \int_{-\infty}^{\infty} \alpha(x) e^{s\delta(x)} dx = \int_{-\infty}^{\infty} \alpha(x) e^{s(\delta(a) - \frac{\delta''(a)}{2}(x-a)^2 + \dots)} dx =$$
$$= e^{s\delta(a)} \int_{-\infty}^{\infty} \alpha(x) e^{-\frac{\delta''(a)}{2}s(x-a)^2 + \dots} dx \sim \frac{e^{s\delta(a)} \cdot \alpha(a) \cdot \sqrt{2\pi}}{\sqrt{-s\delta''(a)}}$$

More formally:

We will find asymptotics for $\int_{-\infty}^{\infty} f(x) s \alpha(x) dx$ for good f, α .

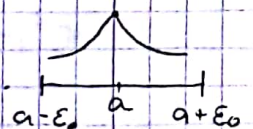
Given $f, a, s_0, k, \epsilon, M$, the class of α we'll look at is:

- 1) $\alpha(a)' \leq k$
- 2) $\sup_{|x-a| < \epsilon_0} \left| \frac{\alpha(x) - \alpha(a)}{x-a} \right| \leq \epsilon$
- 3) $\int f(x) s_0 |\alpha(x)| dx \leq M$

Thm. (Laplace):

Let $f: \mathbb{R} \rightarrow [0, \infty)$ s.t. for some a, ϵ_0, C :

- 1) f is unimodal in the ϵ_0 neighborhood of a .
- 2) For all $\epsilon < \epsilon_0$, $\sup_{|x-a| < \epsilon} f(x) \leq f(a) - C\epsilon^2$. Also $f(x) > 0$ for $|x-a| < 2\epsilon_0$.
- 3) f is twice cont. diff. in $(a-2\epsilon_0, a+2\epsilon_0)$
- 4) $f''(a) < 0$



Then, uniformly in α in the above class:

$$\int f(x) s \alpha(x) dx \sim \sqrt{s} f(a) \sqrt{\frac{2\pi f(a)'}{f''(a)}} \alpha(a)$$

Proof:

$$E(s) := E_0 \cdot \left(\frac{s-s_0}{s}\right)^{1/4} \quad \text{so that} \quad E(s) \in E_0 \quad \text{for } s \geq s_0,$$

$$E(s) \rightarrow 0, \quad \sqrt{s} E(s) \xrightarrow{s \rightarrow \infty} \alpha$$

$$\int f(x)^s \alpha(x) dx = \underbrace{\alpha(a) \int_{|x-a| \leq E(s)} f(x)^s dx}_I + \underbrace{\int_{|x-a| \leq E(s)} f(x)^s (\alpha(x) - \alpha(a)) dx}_II + \underbrace{\int_{|x-a| > E(s)} f(x)^s \alpha(x) dx}_III$$

For I: Write in $|x-a| \leq 2E_0$. $f(x) = f(a) e^{h(x-a)/(x-a)^2}$

for $h(t) = \int_0^1 (1-r) (\log f)''(a+rt) dr$ using int. by parts.

Our assumptions on f give $h(0) = \frac{f''(a)}{2f(a)} < 0$ &

$$e^{h(\frac{t}{\sqrt{s}})t^2} \leq e^{-\frac{ct^2}{f(a)}}$$

$$\text{Then: } I = f(a)^s \int_{|x-a| \leq E(s)} e^{h(x-a)/(x-a)^2} dx \stackrel{y = \sqrt{s}(x-a)}{=} \frac{1}{\sqrt{s}} f(a)^s \int_{|y| \leq \sqrt{s} E(s)} e^{h(\frac{y}{\sqrt{s}})y^2} dy \sim$$

$$\stackrel{s \rightarrow \infty}{\sim} \frac{f(a)^s}{\sqrt{s}} \int_{-\infty}^{\infty} e^{h(0)y^2} dy$$

dominated conv.

$$|III| \leq 2 \int_{|x-a| \leq E(s)} f(x)^s |x-a| dx \leq 2 E(s) \int_{|x-a| \leq E(s)} f(x)^s dx = 2 E(s) \cdot I \ll I$$

$$|III| \leq \int_{|x-a| > E(s)} f(x)^s |\alpha(x)| dx \leq \sup_{|x-a| > E(s)} f(x)^{s-s_0} \cdot M \leq f(a)^{s-s_0} \left(1 - \frac{c E(s)^2}{f(a)}\right)^{s-s_0} M$$

$$\leq e^{-\frac{c E(s)^2}{f(a)}(s-s_0)} M$$

$$\leq e^{-c/\sqrt{s}}$$