

Lecture 7:

5/12/2017

Reminder:

Gaussian Unitary Ensemble (GUE)

$(\xi_{ij}), (\eta_{ij})$ IID $N(0,1)$

$$H_N = \begin{pmatrix} \xi_{11} & \xi_{12} + i\eta_{12} \\ \text{Herm} & \ddots \\ & \xi_{NN} \end{pmatrix}$$

Eigenvalues $\lambda_1^N \leq \dots \leq \lambda_N^N$ are indep. of the eigenvectors.

$$H_N \stackrel{d}{=} U^* H_N U \quad \forall U \text{ unitary.}$$

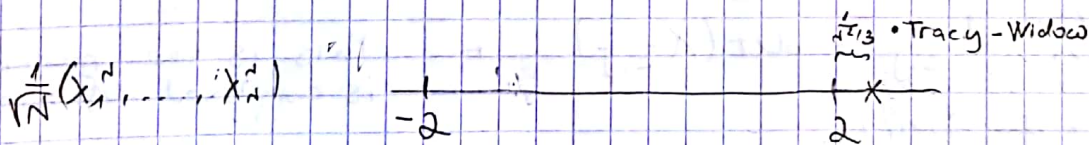
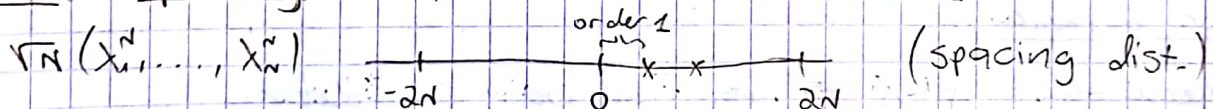
Thm.:

Joint density of $(\lambda_1^N, \dots, \lambda_N^N) \in \mathbb{R}^N$ w.r.t Leb. measure on $\mathbb{R}^N$ is:

$$N! C_N \mathbb{1}_{\lambda_1 \leq \dots \leq \lambda_N} |\Delta(\lambda)|^2 \prod_{i=1}^N e^{-\frac{\lambda_i^2}{2}}; \quad \Delta(\lambda) = \prod_{i < j} (\lambda_j - \lambda_i)$$

$$C_N = \frac{1}{(\sqrt{2\pi})^N \prod_{j=0}^{N-1} j!} \cdot \frac{1}{N!}$$

Goal: (1) spacing dist, dist. of the (2) maximal eigenvalues.



For (1), need to find $\lim_{N \rightarrow \infty} P(\sqrt{N} \lambda_1^N, \dots, \sqrt{N} \lambda_N^N \in [-\frac{t}{2}, \frac{t}{2}])$

Main tool: $(\lambda_1^N, \dots, \lambda_N^N)$ form a determinantal point process.

Probability of interest is a Fredholm determinant.

$\mu_1, \dots, \mu_N$ are the unordered eigenvalues obtained by uniformly permuting $\lambda_1^N, \dots, \lambda_N^N$. Joint density of $(\mu_1^N, \dots, \mu_N^N)$ is:

$$C_N |\Delta(\mu)|^2 \prod_{i=1}^N e^{-\frac{\mu_i^2}{2}}$$

Hermite Polynomials:

These are the orthogonal poly. of the Gaussian density $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$. That is, in $Z_2(R, \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}})$ Consider $1, x, x^2, \dots$ & apply Gram-Schmidt Procedure to get (normalized) Hermite poly. In particular, $h_n(x)$ is a poly. of degree n .

orthogonality: $\int h_k(x) h_l(x) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 0$ when $k \neq l$.

Def.:

The n 'th Hermite poly. is:

$$h_n(x) := (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}} \quad (\text{Rodriguez-type formula})$$

Properties:

(2)

(1) $h_0 \equiv 1$, $h_1(x) = x$. For $n \geq 0$, $h_{n+1}(x) = x h_n(x) - h'_n(x)$ (*)

$\Rightarrow h_n$ is a monic poly. of deg n .

coefficient of x^n is 1.

For (*): $h_{n+1}(x) = ((-1)^n e^{\frac{x^2}{2}} h_n(x))' \cdot (-1)^n e^{\frac{x^2}{2}}$

(3) also, h_n is even or odd according to the parity of n .

(4) $\forall k, l \geq 0$, $\int h_k(x) h_l(x) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = k! \delta_{kl}$

$$(-1)^l \int h_k(x) \frac{d^l}{dx^l} e^{-\frac{x^2}{2}} \frac{1}{\sqrt{2\pi}} x = \int \frac{d^l}{dx^l} h_k(x) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

integration by parts.

If $l > k$ this is zero. For $k=l$, get $k!$ since h_k is monic.

Define also the oscillator wave function:

$$\psi_n(x) = \frac{h_n(x) e^{-\frac{x^2}{4}}}{(\sqrt{2\pi} n!)^{1/2}}$$

then $\int \psi_k(x) \psi_l(x) dx = \delta_{kl}$

Lemma 3.2.2:

For any $1 \leq p \leq N$, the joint density of $\mu_1, \dots, \mu_p$ is:

$$\frac{(N-p)!}{N!} \det_{i,j=1}^p K^{(N)}(x_i, x_j)$$

where $K^{(N)}(x, y) = \sum_{k=0}^{N-1} \Psi_k(x) \Psi_k(y)$

$$\begin{pmatrix} K^{(N)}(x_1, x_1) & K^{(N)}(x_1, x_2) & \dots \\ K^{(N)}(x_2, x_1) & & \\ \vdots & & \end{pmatrix}$$

Proof:

Start with $p=N$. $\Delta(x) = \det \begin{pmatrix} x_1 & \dots & x_n \\ \vdots & & \vdots \\ x_1^{n-1} & \dots & x_n^{n-1} \end{pmatrix} \uparrow \det \begin{pmatrix} 1 & \dots & 1 \\ h_1(x_1) & \dots & h_1(x_n) \\ \vdots & & \vdots \\ h_{n-1}(x_1) & \dots & h_{n-1}(x_n) \end{pmatrix}$

as h_n is a monic poly. of deg. n .

Then $C_N |\Delta x|^2 \prod_{i=1}^N e^{-\frac{x_i^2}{2}} =$

$$= C_N \det \begin{pmatrix} 1 & \dots & 1 \\ h_1(x_1) & \dots & h_1(x_n) \\ \vdots & & \vdots \\ h_{n-1}(x_1) & \dots & h_{n-1}(x_n) \end{pmatrix}^2 \det \begin{pmatrix} e^{-\frac{x_1^2}{2}} & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & e^{-\frac{x_n^2}{2}} \end{pmatrix}^2 =$$

$$= C_N (\sqrt{2\pi})^n \prod_{j=0}^{n-1} j! \det \begin{pmatrix} \Psi_0(x_1) & \dots & \Psi_0(x_n) \\ \Psi_1(x_1) & \dots & \Psi_1(x_n) \\ \vdots & & \vdots \\ \Psi_n(x_1) & \dots & \Psi_n(x_n) \end{pmatrix}^2 = \frac{1}{N!} \det \begin{pmatrix} K^{(N)}(x_1, x_1) & K^{(N)}(x_1, x_2) & \dots \\ K^{(N)}(x_2, x_1) & & \\ \vdots & & \end{pmatrix}$$

$= 1/N!$

multiply transpose of matrix by itself.

How to integrate this density?

Lemma 3.2.3:

Let $f_1, \dots, f_n, g_1, \dots, g_n \in \mathcal{L}_2(\mathbb{R})$,

$$\frac{1}{n!} \iiint \dots \int \det \begin{pmatrix} \sum_{k=1}^n f_k(x_1) g_k(x_1) & \sum_{k=1}^n f_k(x_1) g_k(x_2) & \dots \\ \sum_{k=1}^n f_k(x_2) g_k(x_1) & & \\ \vdots & & \end{pmatrix} dx_1 \dots dx_n \equiv$$

$$\equiv \frac{1}{n!} \int \dots \int \det \begin{pmatrix} f_1(x_1) & \dots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \dots & f_n(x_n) \end{pmatrix} \det \begin{pmatrix} g_1(x_1) & \dots & g_1(x_n) \\ \vdots & \ddots & \vdots \\ g_n(x_1) & \dots & g_n(x_n) \end{pmatrix} dx_1 \dots dx_n$$

$$= \det \begin{pmatrix} \int f_1(x) g_1(x) & \int f_1(x) g_2(x) & \dots \\ \int f_2(x) g_1(x) & \dots & \dots \\ \vdots & \dots & \dots \end{pmatrix}$$

Proof:

$$\mu = \sigma \circ \tau^{-1}$$

$$= \frac{1}{n!} \sum_{\sigma, \tau \in S_n} \text{sgn}(\sigma) \text{sgn}(\tau) \int \dots \int \prod_{i=1}^n f_{\sigma(i)}(x_i) g_{\tau(i)}(x_i) dx_i =$$

$$\prod_{i=1}^n \int f_{\sigma(i)}(x) g_{\tau(i)}(x) dx$$

$$= \sum_{\mu \in S_n} \text{sgn}(\mu) \prod_{i=1}^n f_{\mu(i)}(x) g_i(x) dx = \text{wanted det.}$$

In part. $\iint \dots \int \frac{1}{N!} \det_{i,j=1}^N k^N(x_i, x_j) dx_1 \dots dx_n = 1.$

proceed to $1 \leq p < N$. Need to evaluate:

$$\iint \dots \int \frac{1}{N!} \det \begin{pmatrix} \psi_0(x_1) & \dots & \psi_0(x_n) \\ \vdots & \ddots & \vdots \\ \psi_p(x_1) & \dots & \psi_p(x_n) \end{pmatrix}^2 \prod_{i=p+1}^N dx_i =$$

↑ open det. using permutations.

$$= \frac{1}{n!} \sum_{\sigma, \tau \in S_n} \text{sgn}(\sigma) \text{sgn}(\tau) \int \prod_{i=0}^{p-1} \psi_{\sigma(i)}(x_{i+1}) \psi_{\tau(i)}(x_{i+1}) \prod_{i=p+1}^N dx_i =$$

perm. of $\{0, \dots, n-1\}$

$$= \frac{1}{n!} \sum_{\substack{\sigma, \tau \in S_n \\ \sigma(i) = \tau(i), i \geq p+1}} \text{sgn}(\sigma) \text{sgn}(\tau) \prod_{i=0}^{p-1} \psi_{\sigma(i)}(x_{i+1}) \psi_{\tau(i)}(x_{i+1}) =$$

$$= \frac{1}{N!} \sum_{0 \leq v_1 < v_2 < \dots < v_p \leq N-1} \sum_{\substack{\sigma, \tau \in S_n \\ \sigma(i) = \tau(i), i \geq p+1 \\ \{\sigma(1), \dots, \sigma(p)\} = \{v_1, \dots, v_p\}}} \text{sgn}(\sigma) \text{sgn}(\tau) \prod_{i=0}^{p-1} \psi_{\sigma(i)}(x_{i+1}) \psi_{\tau(i)}(x_{i+1})$$

$$= \frac{1}{N!} \sum_{0 \leq v_1 < v_2 < \dots < v_p \leq N-1} (N-p)! \sum_{\mu: \{0, \dots, p-1\} \rightarrow \{v_1, \dots, v_p\}} \text{sgn}(\mu) \prod_{i=0}^{p-1} \psi_{\mu(i)}(x_{i+1}) \psi_{v_i}(x_{i+1})$$

$$= \frac{(N-p)!}{N!} \sum_{0 \leq v_1 < \dots < v_p \leq N-1} \det \begin{pmatrix} \psi_{v_1}(x_1) & \psi_{v_1}(x_2) & \dots & \psi_{v_1}(x_p) \\ \psi_{v_2}(x_1) & & & \\ \vdots & & & \\ \psi_{v_p}(x_1) & & & \end{pmatrix}^2 =$$

$$= \frac{(N-p)!}{N!} \det \begin{pmatrix} \psi_1(x_1) & \psi_2(x_1) & \dots & \psi_{N-p}(x_1) \\ \vdots & \vdots & & \vdots \\ \psi_1(x_p) & \dots & \dots & \psi_{N-p}(x_p) \end{pmatrix} \begin{pmatrix} \psi_1(x_1) & \dots & \psi_1(x_p) \\ \vdots & & \vdots \\ \psi_{N-p}(x_1) & \dots & \psi_{N-p}(x_p) \end{pmatrix}$$

Cauchy-Binet:

$$P \begin{pmatrix} A \\ B \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix}^P$$

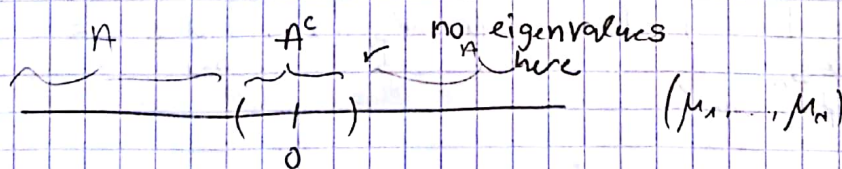
$$\det(G) = \sum_{\substack{I \subseteq \{1, \dots, N\} \\ |I| = p}} \det(A_I) \det(B_I)$$

A_I - columns of A indexed by I

B_I - rows of B indexed by I

$$= \frac{(N-p)!}{N!} \det_{i,j=1}^p K^N(x_i, x_j)$$

Joint density of $\mu_1, \dots, \mu_p$ is $\frac{(N-p)!}{N!} \det_{i,j=1}^p (K^N(x_i, x_j))$



Lemma 3.2.4:

For any Borel $A \subseteq \mathbb{R}$,

$$P(X_1^N \in A, \dots, X_n^N \in A) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int \dots \int \overbrace{\det_{i,j=1}^k K^N(x_i, x_j)}^{k\text{-fold det.}} \prod_{i=1}^k dx_i$$

(same as summing $1 \leq k \leq N$ as the expr is 0 afterward)

$\underbrace{A^c A^c \dots A^c}_{k\text{-fold integral}}$

This is a Fredholm determinant (of $K^{(N)}$ w.r.t. Leb. meas. on K)

Proof:

Probability in question equals $\frac{1}{N!} \int_A \dots \int_A \det(K^{(N)}(x_i, x_j)) \prod_{i=1}^N dx_i$

Lemma 3.23 with $f_i = g_i = \psi_i \mathbb{1}_A$

$$\det \begin{pmatrix} \int_A \psi_1(x) \psi_1(x) dx & \int_A \psi_1(x) \psi_2(x) dx & \dots \\ \int_A \psi_2(x) \psi_1(x) dx & \dots & \dots \\ \vdots & \dots & \dots \end{pmatrix} =$$

$$= \det \begin{pmatrix} I - \int_{A^c} \psi_1(x) \psi_1(x) dx & \int_{A^c} \psi_1(x) \psi_2(x) dx & \dots \\ \int_{A^c} \psi_2(x) \psi_1(x) dx & \dots & \dots \\ \vdots & \dots & \dots \end{pmatrix} \quad (\textcircled{=})$$

$$\det(A+B) = \sum_{I \subseteq \{1, \dots, N\}} \det((A|B)_I)$$

I 'th rows taken from A
 I^c 'th rows taken from B

$$\textcircled{=} 1 + \sum_{k=1}^N \sum_{\substack{J \subseteq \{1, \dots, N\} \\ |J|=k}} (-1)^k \det \begin{pmatrix} \int_{A^c} \psi_{j_1}(x) \psi_{j_1}(x) dx & \int_{A^c} \psi_{j_1}(x) \psi_{j_2}(x) dx & \dots \\ \vdots & \dots & \dots \\ \int_{A^c} \psi_{j_k}(x) \psi_{j_1}(x) dx & \dots & \dots \end{pmatrix}$$

rows taken from second term
 $J = \{j_1, \dots, j_k\}, j_1 < j_2 < \dots < j_k$

$$= 1 + \sum_{k=1}^N \sum_{1 \leq j_1 < \dots < j_k \leq N} \frac{(-1)^k}{k!} \int_{A^c} \dots \int_{A^c} \det \begin{pmatrix} \psi_{j_1}(x_1) & \dots & \psi_{j_k}(x_1) \\ \vdots & \dots & \vdots \\ \psi_{j_1}(x_k) & \dots & \psi_{j_k}(x_k) \end{pmatrix} \prod_{i=1}^k dx_i =$$

Cauchy-Binet

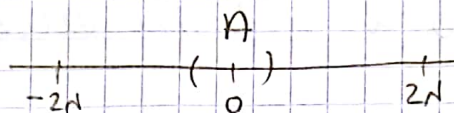
$$= 1 + \sum_{k=1}^N \frac{(-1)^k}{k!} \int_{A^c} \dots \int_{A^c} \det \begin{pmatrix} \psi_1(x_1) & \dots & \psi_{N-k+1}(x_1) \\ \vdots & \dots & \vdots \\ \psi_1(x_k) & \dots & \psi_{N-k+1}(x_k) \end{pmatrix} \left(\begin{smallmatrix} \text{same} \\ \text{transpose} \end{smallmatrix} \right) \prod_{i=1}^k dx_i =$$

$= K^{(N)}(x_i, x_j)$

= Same with $\sum_{k=1}^{\infty}$ as the rank of the product of matrices is $\leq N$.

Spacing dist.:

Thm. (Gaudin - Mehta):



For any compact $A \subseteq \mathbb{R}$, $\lim_{N \rightarrow \infty} P(\sqrt{N}X_1, \dots, \sqrt{N}X_N \notin A) =$

$$= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_A \dots \int_A \det_{i,j=1}^k (S(x_i, x_j)) \prod_{i=1}^k dx_i$$

where S is the Sine Kernel:

$$S(x, y) = \begin{cases} \frac{1}{\pi} \frac{\sin(x-y)}{x-y} & x \neq y \\ \frac{1}{\pi} & x = y \end{cases}$$

First step in proof:

Define "approximation of sine kernel",

$$S^{(N)}(x, y) = \frac{1}{\sqrt{N}} K^{(N)}\left(\frac{x}{\sqrt{N}}, \frac{y}{\sqrt{N}}\right)$$

Main Lemma:

$\lim_{N \rightarrow \infty} S^{(N)}(x, y) = S(x, y)$ uniformly for x, y in a compact

subset A of $\mathbb{R}$.

Proof of thm. with main lemma & continuity:

$$P(\sqrt{N}X_1, \dots, \sqrt{N}X_N \notin A) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{A/\sqrt{N}} \dots \int_{A/\sqrt{N}} \det_{i,j=1}^k K^{(N)}(x_i, x_j) \prod_{i=1}^k dx_i$$

divide each $\hat{x}_i$ by $\sqrt{N}$

$$= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_A \dots \int_A \det_{i,j=1}^k S^{(N)}(x_i, x_j) \prod_{i=1}^k dx_i \xrightarrow[\text{continuity}]{\text{same with } S^{(N)}} 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_A \dots \int_A \det_{i,j=1}^k S(x_i, x_j) \prod_{i=1}^k dx_i$$