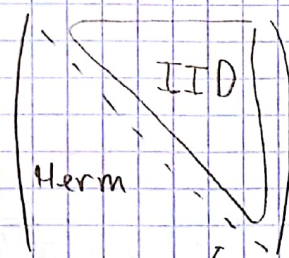


Gaussian Orthogonal Unitary ensemble (GOE, GUE):

real symmetric

Hermitian

IID on diagonal.

start with $(\xi_{i,j}), (\eta_{i,j})$ IID $N(0,1)$ $N(0,2)$

indep.

$$\text{GOE} : \begin{pmatrix} \sqrt{2}\xi_{11} & \xi_{12} \\ \xi_{12} & \sqrt{2}\xi_{22} \end{pmatrix} (n=2), \begin{pmatrix} \sqrt{2}\xi_{11} & \xi_{12} & \xi_{13} \\ \xi_{12} & \sqrt{2}\xi_{22} & \xi_{23} \\ \xi_{13} & \xi_{23} & \sqrt{2}\xi_{33} \end{pmatrix} (n=3)$$

 $N(0,1)$ $N_{\mathbb{C}}(0,1)$

$$\text{GUE} : \begin{pmatrix} \xi_{11} & \frac{\xi_{12} + i\eta_{12}}{\sqrt{2}} \\ \frac{\xi_{12} - i\eta_{12}}{\sqrt{2}} & \xi_{22} \end{pmatrix} (n=2) \quad \mathbb{E} |N_{\mathbb{C}}(0,1)|^2 = 1$$

Main feature: Eigenvalues & eigenvectors are indep.Density of GOE/GUE on matrices:Reference measure:Denote by $M_N^{(\beta)}$ the set of: $\beta=1$: real symmetric matrices $\beta=2$: complex Hermitian matrices.Denote by $\mu_N^{(\beta)}$ the "Lebesgue" measure on $M_N^{(\beta)}$ defined by: $\beta=1$: Product Lebesgue on diagonal & above diagonal entries. $\beta=2$: Product Lebesgue on diagonal & on real & imaginary parts of each above diag. entry.

$$\dim(\mu_N^{(1)}) = \binom{N+1}{2}, \quad \dim(\mu_N^{(2)}) = N^2.$$

For GOE, joint density is:

$$\prod_{i=1}^N \frac{1}{\sqrt{2\pi}} e^{-\frac{X_{ii}^2}{2}} \prod_{j>i} \frac{1}{\sqrt{2\pi}} e^{-\frac{X_{ij}^2}{2}} d\mathbf{X}^{(N)} = \frac{1}{(2\pi)^{\frac{N(N+1)}{2}}} e^{-\frac{1}{4} \sum_{i,j} X_{ij}^2} = e^{-\frac{1}{4} \text{Tr}(X^2)}$$

Density of X is a function only of the eigenvalues of X . For GUE, joint density = $\frac{1}{2^{N/2}} \cdot \frac{1}{\pi^{N/2}} e^{-\frac{1}{2} \text{Tr}(X^2)}$

Can write $X = U D U^*$

$\beta=1$: Orthogonal
 $\beta=2$: Unitary
columns are the
eigenvectors.

diagonal, eigenvalues
on diagonal

"When you condition on the eigenvalues, the eigenvectors are a uniformly picked (Haar measure) orthonormal basis."

Main theorem of today: Joint density of the eigenvalues
 $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$

Def:

For $x \in \mathbb{C}^N$, $\Delta(x) = \prod_{i<j} (x_i - x_j) = \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_N \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{N-1} & x_2^{N-1} & \dots & x_N^{N-1} \end{pmatrix}$

Thm:

Let X be GOE / GUE distributed. The joint dist. of $(\lambda_1, \lambda_2, \dots, \lambda_N)$ is a density w.r.t. Leb. meas. on $\mathbb{R}^N$, given by:

$$N! C_N^{(\beta)} |\Delta(x)|^\beta \prod_{i=1}^N e^{-\beta \frac{X_i^2}{2}} \mathbb{1}_{\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N} \prod_{i=1}^N dx_i$$

$$= \frac{1}{(2\pi)^{N/2}} \left(\frac{\beta}{2}\right)^{\frac{N}{2} + \frac{\beta}{2} \binom{N}{2}} \prod_{j=1}^N \frac{\Gamma(\beta/2)}{\Gamma(j\beta/2)}$$

Selberg
integral

where $\Gamma(s) := \int_0^\infty x^{s-1} e^{-x} dx$

Gamma function

$\rightarrow \Gamma(n+1) = n!$ for $n \geq 0$ integer.

$\rightarrow \Gamma(\frac{1}{2}) = \sqrt{\pi}$

$\rightarrow \Gamma(x+1) = x \Gamma(x)$

Idea of Proof: $X = UDU^*$

$\uparrow$ $\uparrow$
 orthogonal ($\beta=1$) diagonal, eigenvalues
 unitary ($\beta=2$)
 eigenvectors

Study the map $(D, U) \longrightarrow UDU^*$, analyze what is the push-forward through it of a dist. on D .

Jacobian of the map will equal $G(U) \Delta(x)^\beta$ where x is the diagonal of D .

This will not give us the value of $G_N^{(\beta)}$. we will not calculate it now. Later, for $\beta=2$.

Example: $\begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix}$ - orthogonal - 3 real degrees of freedom

More generally: - orthogonal - $\binom{N}{2}$ degrees of freedom

unitary - $2\binom{N}{2}$ degrees of freedom.

Proof:

Def. of desirable properties of D, U :

- U is normalized if all diagonal entries are real & strictly positive.
- U is good if it is normalized & has no zero entries.
- U is very good if it is good & all its minors are non-zero
 $\left(\text{minor} = \det(U_{I,J}) \text{ where } I, J \subseteq \{1, \dots, N\}, |I| = |J| > 0 \right)$
 $U_{I,J}$ - I 'th rows of U
 J 'th columns of U
- D is distinct if all diagonal entries are distinct.
- D is ordered if diagonal entries are decreasing (& distinct).
- $M_N^{(\beta)}$ - set of all $\begin{matrix} \beta=1 : \text{real symmetric} \\ \beta=2 : \text{Hermitian} \end{matrix}$ matrices Endowed with Leb. measure $d_N^{(\beta)}$

• $M_N^{(\beta), dg} = \{X \in M_N^{(\beta)} \mid X = UDU^*, D \text{ distinct, } U \text{ good}\}$

Lemma 1:

$$l_N^{(\beta)}(M_N^{(\beta)} \setminus M_N^{(\beta), dg}) = 0$$

Additionally, the map $(D, U) \rightarrow UDU^*$ is one to one & onto from $(\underset{\substack{\uparrow \\ \text{distinct-ordered}}}{D}^{\text{do}}, \underset{\substack{\uparrow \\ \text{good}}}{U}^g)$ to $M_N^{(\beta), dg}$

(& $N!$ to 1 if D is not ordered)

Lemma 2:

distinct, very good

$$l_N^{(\beta)}(M_N^{(\beta)} \setminus M_N^{(\beta), dv}) = 0$$

Lemma 3: (smooth parametrization of U):

The map $T: U^{(\beta), vg} \rightarrow \mathbb{R}^{\beta(N/2)}$ given by:

$$T(U) = \left(\frac{U_{11}}{U_{11}}, \dots, \frac{U_{1n}}{U_{1n}}; \frac{U_{23}}{U_{22}}, \dots, \frac{U_{2n}}{U_{22}}; \dots; \frac{U_{n+1,n}}{U_{n+1,n}}, \frac{U_{n+1,n}}{U_{n+1,n}} \right)$$

is one to one with smooth inverse.

Further, the set $T(U^{(\beta), vg})^c$ is closed with zero Lebesgue measure.

This gives a map from $\mathbb{R}^n \times \mathbb{R}^{\beta(N/2)}$ to $M_N^{(\beta), dv}$ by:

$$(x, z) \xrightarrow{\hat{T}} T^{-1}(z) D_x T^{-1}(z)^* \quad \begin{array}{l} \text{diag values} \uparrow \quad \downarrow T(U) \\ \text{taking } x \text{ distinct \& } z \in T(U^{(\beta), vg}) \\ \text{diagonal matrix with } x \text{ on diagonal} \end{array}$$

Proof of thm. from lemmas:

Analyze Jacobian of $\hat{T}$.

$$J\hat{T} = \det \left(\frac{\partial \hat{T}_{ij}}{\partial x_k}, \frac{\partial \hat{T}_{ij}}{\partial z_{m,j}} \right). \text{ By def. } \left(\hat{T}_{ij} = T^{-1}(z) D_x T^{-1}(z)^* \right) = \sum_{k=1}^n c_{kj}^{ij}(z) x_k$$

is linear in x , with coefficients depending on z .

So, $\frac{\partial \hat{T}_{ij}}{\partial x_k}$ is a fcn only of z , $\frac{\partial \hat{T}_{ij}}{\partial z_{m,j}}$ is linear in x , with

coef. depending on z .

Conclude that $\text{Jac}(\hat{T})$ is a polynomial of $\deg \leq \beta \binom{N}{2}$ in x , which is homogeneous (or the zero polynomial).

real case:

$$\text{Jacobi matrix} = \begin{matrix} N \\ \left(\begin{matrix} N \\ 2 \end{matrix} \right) \end{matrix} \left\{ \begin{array}{l} \text{do not depend on } x \\ \vdots \\ \text{Depend linearly on } x \end{array} \right\} \begin{matrix} \frac{\partial T_{ij}}{\partial x_1} \\ \vdots \\ \frac{\partial T_{ij}}{\partial x_N} \\ \frac{\partial T_{ij}}{\partial z_{m+1}} \end{matrix}$$

not justified & not used $\left\{ \text{Jac}(\hat{T}) \neq 0 \text{ on } M_N^{(\beta)} \text{ dvg} \right.$ Since $\hat{T}$ is invertible in a small neighborhood of each matrix there. (say, to ordered eigenvalues).

Implicit fcn. thm: $\text{Jac}(\hat{T}) = 0$ if $x_i = x_j$ for some $i \neq j$ since $\hat{T}$ is non-invertible in a neigh of such points.

It follows that $\text{Jac}(\hat{T})$ is a multiple of $\Delta(x)$ (by some poly. in x with coef. depending on z).

In fact, $\text{Jac}(\hat{T})$ is divisible by $\Delta(x)^\beta$. It follows, since $\text{Jac}(\hat{T})$ is a poly. of $\deg \beta \binom{N}{2}$ in x , that

$$\text{Jac}(\hat{T}) = \Delta(x)^\beta \cdot G(z).$$

Now take f a function of the eigenvalues of X .

$$E(f(X)) = \int f(\lambda_1, \dots, \lambda_n) \text{ density of eigenvalues}$$

$$E(f(X)) = \int_{M_N^{(\beta)}} f(\lambda_1(X), \dots, \lambda_n(X)) \cdot C_\beta e^{-\frac{\beta}{4} \text{tr}(X^2)} d\ell_N^{(\beta)}(X)$$

$$= \int_{M_N^{(\beta)}} f(\lambda_1(X), \dots, \lambda_n(X)) C_\beta e^{-\frac{\beta}{4} \sum_{i=1}^n \lambda_i(X)^2} d\ell_N^{(\beta)}(X) \stackrel{\text{Jacobian Calculation}}{=} \int_{M_N^{(\beta)}} f(x_1, \dots, x_n) C_\beta e^{-\frac{\beta}{4} \sum_{i=1}^n x_i^2} \Delta(x)^\beta G(z) dx dz =$$

$$= \int C_\beta G(z) dz \int_{x_1 \leq \dots \leq x_n} f(x_1, \dots, x_n) e^{-\frac{\beta}{4} \sum_{i=1}^n x_i^2} \Delta(x)^\beta dx$$

normalization const.

density of eigenvalues

Why $\Delta(x)^2$ when $\beta=2$ divides $\text{Jac}(\hat{f})$?

For brevity, write $\hat{T}(x, z) = W D W^*$

$T^{-1}(z)$

Differential:

$$d\hat{T} = (dW) D W^* + W (dD) W^* + W D (dW^*) \quad (*)$$

$$\frac{\partial \hat{T}_{ij}}{\partial z_{m,l}} = \left(\frac{\partial W}{\partial z_{m,l}} D W^* + \dots \right)_{ij} \leftarrow \text{explanation of } d\hat{T}.$$

Now, as $W W^* = I$, whence

$$0 = d(W W^*) = (dW) W^* + W (dW^*) \quad (**)$$

$$(*) + (**) \Rightarrow$$

$$\begin{aligned} W^* d\hat{T} W &= W^* (dW) D + dD + D (dW^*) W = \quad (*) \\ &= dD + W^* (dW) D - D W^* (dW) \end{aligned}$$

Now suppose that $x_i = x_j$ for some $i \neq j$.

$$\underbrace{W^* (dW) D}_{\text{multiplying columns by entries of } D} - \underbrace{D W^* (dW)}_{\text{multiplying rows by entries of } D} = i \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

multiplying columns
by entries of D

multiplying rows
by entries of D

$$\text{Thus } \left(\frac{\partial \text{Re } \hat{T}_{ij}}{\partial z_{m,l}} \right)_{m,l} = \left(\frac{\partial \text{Im } (\hat{T}_{i,j})}{\partial z_{m,l}} \right)_{m,l} = 0$$

Proof of Lemma 1:

Claim:

Let P be a poly. in n variables. Then either P is identically ^{zero} or $\underbrace{\{x \mid P(x)=0\}}_{\text{closed}} = Z$ has zero Leb measure in $\mathbb{R}^n$.

Idea of Proof:

$$\text{Induction on } n. \quad \text{Leb}(Z) = \int dx_1 \int dx_2 \dots \int dx_n \mathbb{1}_Z(x_1, \dots, x_n) > 0$$

Distinct eigenvalues: The discriminant of the characteristic polynomial of $X \in M_n^{(\mathbb{R})}$ is a poly. in the entries of X which vanishes iff the char poly. had a double root, i.e., non distinct eigenvalues.

Goads

Now suppose all eigenvalues of X have multiplicity 1.

Let λ be an eigenvalue of X .

Define $A = X - \lambda I$. Adjoint: $A_{ij}^{adj} = (-1)^{i+j} \det(A^{i,j})$ erase i -th row & j -th column.

$\Rightarrow A A^{adj} = \det(A) I = 0 \Rightarrow$ Each column of A^{adj} is a multiple of the unique eigenvector of λ .

$\Rightarrow$ if all diagonal entries of A^{adj} are non-zero, then the eigenvector of λ has no zero entries.

Now, $A_{i,i}^{adj} = \det(A^{i,i}) = \det((X - \lambda I)^{i,i}) = \det(X^{i,i} - \lambda I)$

Conclusion: If there exists some eigenvalue λ whose eigenvector has a zero entry then the char poly of X & the char poly of $X^{i,i}$, for some i , have a common root.

Use that given two poly. P, Q , there exists another poly.

$r(P, Q)$, the resultant of P, Q , whose coef are poly. in the coef of P, Q which vanishes iff P, Q have a common root.



for Lemma 2:

check that if X is any matrix, can define:

$$\underbrace{(A^r X)}_{\text{matrix of dim } \binom{N}{r} \times \binom{N}{r}}_{I,J} = \det(X_{I,J}) \quad \leftarrow \text{this is the matrix which is composed of minors}$$

$$I, J \subseteq \{1, \dots, N\}, |I| = |J| = r$$

$$\text{Then } (A^r X^{(t)}) = (A^r X)^{(t)*}$$

$$A^r(XY) = \underbrace{(A^r X)(A^r Y)}_{\text{Cauchy-Binet}}$$

orthogonal / unitary

$$\Rightarrow \text{For herm. } X, X = U D U^*. \quad A^r X = (A^r U) \underbrace{(A^r D)}_{\text{diagonal}} (A^r U^*)$$

$\uparrow$
gives us access to the minors.