

Wigner matrices:

$$X_N = \frac{1}{\sqrt{N}} \begin{pmatrix} Y_1 & Z_{12} & \dots \\ Y_2 & Y_2 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

$(Y_i), (Z_{ij})$ - IID
indep.

$$\mathbb{E}(Y_i) = \mathbb{E}(Z_{ij}) = 0, \quad \mathbb{E} Z_{12}^2 = 1$$

Gaussian Wigner: $Y_i \sim N(0, \sigma^2)$

$$Z_{ij} \sim N(0, 1)$$

Eigenvalue: $\lambda_1^N \leq \dots \leq \lambda_N^N$

$$Z_N = \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i^N}, \quad \bar{Z}_N = \mathbb{E} Z_N$$

for any $f: \mathbb{R} \rightarrow \mathbb{R}$ suitably integrable ($[0,1]$ -meas.)

$$\int f(x) d\bar{Z}_N(x) = \langle \bar{Z}_N, f \rangle = \mathbb{E} \langle Z_N, f \rangle = \mathbb{E} \left(\int f(x) dZ_N(x) \right)$$

Functions of sym or herm. matrices:

for H herm. & $f: \mathbb{R} \rightarrow \mathbb{R}$, let $f(H)$ be defined as follows:

It has the same eigenvectors as H & an eigenvalue λ becomes

$f(\lambda)$. I.e., $H = U D U^*$ for a unitary U & diagonal D

$$\& f(H) = U f(D) U^*$$

$$\begin{pmatrix} f(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & f(\lambda_N) \end{pmatrix}$$

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix}$$

Note that this is consistent with def. for $f(x) = x^k$, $k=1$ integer

& also $f = \frac{\text{poly.}}{\text{poly.}}$

Concentration ineq.:

Suppose X_N has indep. entries $X_N(i,j)$, $j \geq i$ & is Herm.

Suppose that the distribution of each $X_N(i,j)$ satisfies

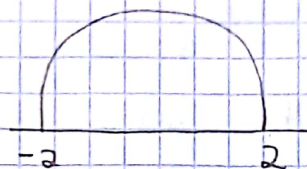
the LSI with const. $\frac{c}{N}$ (e.g., Gaussian Wigner,

$c = \max(\sigma^2, 1)$) & $f: \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz. Then:

$$P \left(\left| \underbrace{\frac{1}{N} \text{tr}(f(X_N))}_{= \langle Z_N, f \rangle} - \underbrace{\mathbb{E} \left[\frac{1}{N} \text{tr}(f(X_N)) \right]}_{= \langle \bar{Z}_N, f \rangle} \right| \geq \delta \right) \leq 2e^{-\frac{N^2 \delta^2}{4c N^2 \log p}}$$

Wigner's semicircle thm:

$$\sigma(x) = \frac{1}{2\pi} \sqrt{4-x^2} \mathbb{1}_{|x| \leq 2}$$



Thm:

For any Wigner (with two finite moments) & $f: \mathbb{R} \rightarrow \mathbb{R}$ bdd. cont.,

$$\forall \epsilon > 0 \quad P(|\langle Z_N, f \rangle - \langle \sigma, f \rangle| > \epsilon) \xrightarrow{N \rightarrow \infty} 0$$

(Weak conv. in prob.)

Alternative approach to thm:

Stieltjes transform:

Let μ be a positive finite measure on $\mathbb{R}$. The Stieltjes transform of μ is $S_\mu(z) = \int \frac{\mu(dx)}{x-z}$, $z \in \mathbb{C} \setminus \mathbb{R}$.

$$\text{This converges as: } \langle \mu, \frac{1}{x-z} \rangle$$

This converges as:

$$\left| \frac{1}{x-z} \right| = \frac{1}{|x-z|} \leq \frac{1}{|\text{Im}(z)|}$$

$$\frac{1}{N} \text{Tr}((X_N - zI)^{-1}) = \langle Z_N, \frac{1}{x-z} \rangle$$

The Stieltjes transform of $\sigma(x)dx$ is

$$S_\sigma(z) = \frac{z}{2} \left(\sqrt{1 - \frac{4}{z^2}} - 1 \right) \quad z \in \mathbb{C} \setminus [-2, 2]$$

$$= \frac{\sqrt{z^2 - 4} - z}{2}$$

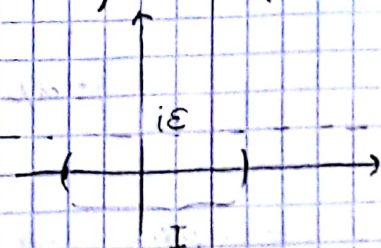
S_μ determines μ .

Thm:

For any open interval $I \subseteq \mathbb{R}$ with no atom of μ at endpoints,

$$\mu(I) = \lim_{\epsilon \downarrow 0} \frac{1}{\pi} \int_I \text{Im}(S_\mu(\lambda + i\epsilon)) d\lambda$$

$$\text{Im}\left(\frac{1}{x - (\lambda + i\epsilon)}\right) = \frac{\epsilon}{(x - \lambda)^2 + \epsilon^2} > 0$$



Proof:

Note $\mu = 0$ iff $S_\mu \equiv 0$ since, e.g., $\text{Im}(S_\mu(i)) = \int \frac{1}{x^2+1} d\mu(x) = 0$
 $\Leftrightarrow \mu = 0$

Can recover $\int \mu(dx)$ since:

$$y \text{Im}(S_\mu(iy)) = \int \frac{y^2}{x^2+y^2} d\mu(x) \xrightarrow{y \rightarrow \infty} \int d\mu(x) \text{ bdd conv. thm.}$$

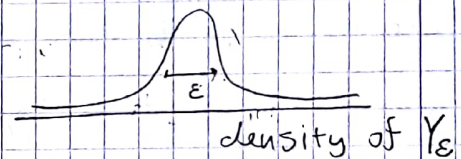
Assume WLOG, μ is a prob. measure.

Suppose X is a RV, $X \sim \mu$.

Observe that: $\frac{1}{\pi} \text{Im}(S_\mu(\lambda + i\varepsilon)) = \frac{1}{\pi} \int \frac{\varepsilon}{(x-\lambda)^2 + \varepsilon^2} d\mu(x)$

which is the density at λ of $X + Y_\varepsilon$ where Y_ε is indep of X & has the Cauchy dist.

$$\text{density}(Y_\varepsilon) = \frac{\varepsilon}{\pi(x^2 + \varepsilon^2)}$$



Remains to note that $X + Y_\varepsilon \xrightarrow{d} X$ as $\varepsilon \downarrow 0$

(see, e.g., by Fourier transform - $\text{FT}(Y_\varepsilon) \propto e^{-\varepsilon|t|}$)

This implies the thm.

Thm: (Relations with conv. of measures):

Let (μ_n) be a sequence of prob measures on $\mathbb{R}$.

✓ 1) If $\mu_n \xrightarrow{d} \mu$ then $S_{\mu_n}(z) \rightarrow S_\mu(z) \quad \forall z \in \mathbb{C} \setminus \mathbb{R}$

$\int f d\mu_n \rightarrow \int f d\mu \quad \forall \text{ bdd. cont. } f$ (follows from definition)

2) Suppose $S_{\mu_n}(z) \rightarrow S(z) \quad \forall z \in \mathbb{C} \setminus \mathbb{R}$, then $S(z)$ is the Stieltjes trans. of a sub-prob. meas. μ ($\int \mu(dx) \leq 1$) & $\mu_n \rightarrow \mu$ Vaguely.

$\int f d\mu_n \rightarrow \int f d\mu$ for any $f: \mathbb{R} \rightarrow \mathbb{R}$, bdd. cont. & Vanishing at infinity.

Example: $\mu_n = \delta_n$

$\mu_n \rightarrow 0$ Vaguely. $S_{\mu_n}(z) = \frac{1}{n-z} \rightarrow 0 \quad \forall z \in \mathbb{C} \setminus \mathbb{R}$

3) Suppose μ_n are random & $S_{\mu_n}(z) \xrightarrow{\text{Prob.}} S_{\mu}(z) \quad \forall z \in \mathbb{C} \setminus \mathbb{R}$
 where μ is a deterministic prob. meas.. Then:

$$\mu_n \longrightarrow \mu \text{ in distribution in Prob.}$$

$$\left(\left| \int f d\mu_n - \int f d\mu \right| > \varepsilon \right) \xrightarrow{n \rightarrow \infty} 0 \text{ for any } f \text{ bdd. cont.}$$

Proof:

2) Helly's selection principle:

If μ_n is a seq. of Prob. meas. then

$\exists \mu$ sub-Prob. meas., n_{k_j} with: $\mu_{n_{k_j}} \longrightarrow \mu$ Vaguely.

Vague convergence is metrizable. Thus enough that $\exists \mu$ sub-Prob. meas.

for any $n_k \exists n_{k_j}$ with $\mu_{n_{k_j}} \longrightarrow \mu$ Vaguely.

Indeed, for each $n_k \exists n_{k_j}$ & μ s.t. $\mu_{n_{k_j}} \longrightarrow \mu$ Vaguely.

whence $S_{\mu_{n_{k_j}}}(z) \longrightarrow S_{\mu}(z) \quad \forall z \in \mathbb{C} \setminus \mathbb{R}$ which means that μ

is the same for all subsequences.

3) Idea (something in exercise):

Let d be a metric on Prob. measures s.t. if $d(\mu_n, \mu) \xrightarrow{d} 0$
 then $\mu_n \longrightarrow \mu$ in dist.

Then, if μ_n is a random seq. of Prob. measures &

$P(d(\mu_n, \mu) > \varepsilon) \xrightarrow{n \rightarrow \infty} 0 \quad \forall \varepsilon > 0$ Then $\mu_n \longrightarrow \mu$ weakly
 (in dist) in Prob.

Let us build a suitable metric d . Let $z \in \mathbb{C} \setminus \mathbb{R}, (z_n) \subseteq \mathbb{C} \setminus \mathbb{R}$,

$z_n \longrightarrow z, z_n \neq z$.

Define $d(\mu_k, \nu) = \sum_{i=1}^{\infty} 2^{-i} |S_{\mu_k}(z_i) - S_{\nu}(z_i)|$

$d(\mu_k, \nu) \longrightarrow 0$ iff $\forall n, S_{\mu_k}(z_n) \longrightarrow S_{\nu}(z_n)$

When this happens $\mu_k \longrightarrow \nu$ in dist. Since any subseq. of μ_k
 has a further subseq. conv. vaguely to some limit, but that
 limit must be ν , since its Stieltjes coincide with ν of (z_n) &
 thus everywhere by analyticity.

Remark: $S_{\mu}(z) = \int \frac{d\mu(x)}{x-z}$, $S_{\mu}(\bar{z}) = \overline{S_{\mu}(z)}$

Our assumption implies that $\rho(d(\mu_n, \mu) > \varepsilon) \xrightarrow{n \rightarrow \infty} 0$

Wigner's thm via Stieltjes transform for Gaussian Wigner matrices

Goal: X_N Gaussian Wigner.

Show: $S_{Z_N}(z) \xrightarrow{\text{prob}} S_{\infty}(z) = \frac{z}{2} \left(\sqrt{1 - \frac{4}{z^2}} - 1 \right) \quad \forall z \in \mathbb{C} \setminus \mathbb{R}$.

Convenient to start with $S_{Z_N}(z) = \mathbb{E} S_{Z_N}(z)$.

Two tricks:

1) For an Herm. matrix X , define:

$$R_X(z) = S_X(z) := (X - zI)^{-1} \quad (\text{resolvent of } X)$$

Identity: $R_X(z) = \frac{1}{z} (X R_X(z) - I)$, $\forall z \in \mathbb{C} \setminus \mathbb{R}$

2) If $\xi \sim N(0, \sigma^2)$, RV & $f: \mathbb{R} \rightarrow \mathbb{R}$ cont. diff, of poly. growth of f, f' , then:

$$\mathbb{E}(\xi f(\xi)) = \sigma^2 \mathbb{E}(f'(\xi))$$

Proof:

$$\begin{aligned} \mathbb{E}(\xi f(\xi)) &= \int x f(x) \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} dx = \int f'(x) \frac{\sigma^2}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} dx \\ &\quad \text{int. by parts} \\ &\quad f \rightarrow f' \\ &\quad x \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \rightarrow -\frac{\sigma^2}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \end{aligned}$$

Calculation:

What is the derivative of $R_X(z)$, with respect to X ?

Real symmetric case $\frac{\partial}{\partial X} R_X(z) = -R_X(z) \Delta^{i,k} R_X(z)$ means that we perturb X by adding ε to $X_{i,k}$ & $X_{k,i}$

$$\Delta^{i,k} = \begin{cases} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \text{pos}(i,k) \quad i \neq k \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \text{pos}(i,i) \quad i = k \end{cases}$$

$$\begin{aligned} \text{Indeed, if } E \text{ is hermitian, then } R_{X+\varepsilon E}(z) &= (X + \varepsilon E - zI)^{-1} = \\ &= [(X - zI) (I + (X - zI)^{-1} \varepsilon E)]^{-1} = (I + \varepsilon (X - zI)^{-1} E)^{-1} R_X(z) \\ &= R_X(z) - \varepsilon R_X(z) E R_X(z) + O(\varepsilon^2) \end{aligned}$$

$I - \varepsilon (X - zI)^{-1} E + \varepsilon^2 (X - zI)^{-1} E^2 + \dots$

$$S_{\bar{z}_n}(z) = \langle \bar{z}_n, \frac{1}{x-z} \rangle = \mathbb{E} \langle \bar{z}_n, \frac{1}{x-z} \rangle = \frac{1}{N} \mathbb{E} \text{tr} [R_{X_n}(z)] =$$

$$= \frac{1}{N} \mathbb{E} \text{tr} \left[\frac{1}{z} (X_n R_{X_n}(z) \cdot I) \right] = -\frac{1}{z} + \frac{1}{Nz} \underbrace{\mathbb{E} \text{tr} (X_n R_{X_n}(z))}_{\text{use trick 2}}$$

Write $X_n = \sum_{\substack{i,k=1 \\ k \geq i}}^N X_n(i,k) \Delta^{i,k}$

$$X_n R_{X_n}(z) = \sum_{\substack{i,k=1 \\ k \geq i}}^N X_n(i,k) \Delta^{i,k} R_{X_n}(z) \stackrel{\text{trick (2)}}{=} \sigma^2 \mathbb{E}(f)$$

$$\mathbb{E} \text{tr} (X_n R_{X_n}(z)) = \sum_{\substack{i,k=1 \\ k \geq i}}^N \underbrace{\mathbb{E} [X_n(i,k) \text{tr} (\Delta^{i,k} R_{X_n}(z))]}_{= f(X_n(i,k)) \text{ with all other entries of } X_n \text{ fixed.}}$$

$$\sigma^2 = \begin{cases} \frac{1}{N} & i \neq k \\ \frac{\mathbb{E}(Y_i^2)}{N} & i = k \end{cases}$$

Conclude:

$$S_{\bar{z}_n}(z) = \frac{1}{z} - \frac{1}{Nz} \mathbb{E} \left[\sum_{i,k=1}^N R_{X_n}(z)(i,i) R_{X_n}(z)(k,k) + R_{X_n}(z)(i,k)^2 \right] -$$

$$- \frac{1}{Nz} \sum_i (\mathbb{E}(Y_i^2) - 2) \mathbb{E}(R_{X_n}(z)(i,i)^2)$$

$$\sum_{i,k} R_{X_n}(z)(i,i) R_{X_n}(z)(k,k) = \left(\sum_i R_{X_n}(z)(i,i) \right)^2 = (\text{tr} (R_{X_n}))^2 =$$

$$= \left(\langle \bar{z}_n, \frac{1}{x-z} \rangle N \right)^2$$

$$\sum_{i,k} R_{X_n}(z)(i,k)^2 = \text{tr} (R_{X_n}(z) \underbrace{R_{X_n}^T(z)}_{= R_{X_n}(z)}) = \text{tr} (R_{X_n}^2(z)) = \langle \bar{z}_n, \frac{1}{(x-z)^2} \rangle N$$

Thus, can put $\bar{z}_n$ by concentration.

$$S_{\bar{z}_n}(z) = -\frac{1}{z} - \frac{1}{z} \mathbb{E} \left(\langle \bar{z}_n, \frac{1}{x-z} \rangle^2 \right) - \frac{1}{Nz} \underbrace{\langle \bar{z}_n, \frac{1}{(x-z)^2} \rangle}_{\substack{O(1) \\ \text{since } \frac{1}{x-z} \text{ is bdd in } X \\ \text{when } z \in \mathbb{R}}} - \frac{1}{Nz} \underbrace{\sum_i (\mathbb{E}(Y_i^2) - 2) \mathbb{E}(R_{X_n}^2(z)(i,i))}_{= O(1) \text{ since } \frac{1}{N} \sum_{i,k} \mathbb{E}(R_{X_n}(z)(i,k)^2) = 0}$$

$$S_{\bar{z}_n}(z) = -\frac{1}{z} - \frac{S_{\bar{z}_n}(z)^2}{z} + o(1) \text{ as } n \rightarrow \infty$$

We conclude that:

$$S_{\bar{z}_n}(z) = -\frac{1}{z} - \frac{S_{\bar{z}_n}(z)^2}{z} + o(1) \text{ as } n \rightarrow \infty.$$

The limit equation $S(z) = -\frac{1}{z} - \frac{S(z)^2}{z}$ which means

$$S(z) = \frac{z}{2} \left(\pm \sqrt{1 - \frac{4}{z^2}} - 1 \right)$$

We have the + sign since $\text{Im}(S_{\bar{z}_n}(\lambda + i\varepsilon)) > 0, \varepsilon > 0,$

We deduce that $S_{\bar{z}_n}(z) \rightarrow \frac{z}{2} \left(\sqrt{1 - \frac{1}{z^2}} - 1 \right)$ for any $z \in \mathbb{C} \setminus \mathbb{R}$.

This implies that $S_{\bar{z}_n}(z) \xrightarrow{P} \frac{z}{2} \left(\sqrt{1 - \frac{1}{z^2}} - 1 \right)$ since $S_{\bar{z}_n}(z) = \langle \bar{z}_n, \frac{1}{z-z} \rangle$ is close in prob. to $S_{\bar{z}_n}(z) = \langle \bar{z}_n, \frac{1}{z-z} \rangle$ by concentration ineq.