

Reminder:

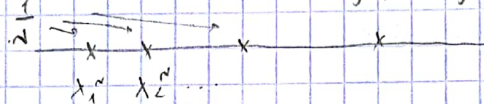
the wigner model:

$$X_N = \frac{1}{\sqrt{N}} \begin{pmatrix} Y_1 & Z_{1,2} & Z_{1,3} & \dots \\ & Y_2 & Z_{2,3} & \dots \\ & & \ddots & \ddots \\ & & & Y_N \end{pmatrix} \quad \left. \vphantom{\begin{pmatrix} Y_1 & Z_{1,2} & Z_{1,3} & \dots \\ & Y_2 & Z_{2,3} & \dots \\ & & \ddots & \ddots \\ & & & Y_N \end{pmatrix}} \right\}^N$$

$$\begin{aligned} (Z_{i,j})_{j \geq i} & \text{ IID} \\ (Y_i) & \text{ IID} \end{aligned} \quad \swarrow \text{ indep.}$$

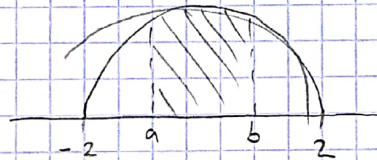
we want to study the eigenvalues of X_N $\lambda_1^N \leq \lambda_2^N \leq \dots \leq \lambda_N^N$

empirical distribution: $Z_N := \frac{1}{N} \sum_{j=1}^N \delta_{X_j^N}$ - this is a measure
atoms of size $\frac{1}{N}$



Wigner's semicircle distribution:

$$\sigma(x) dx \quad \text{where} \quad \sigma(x) = \frac{1}{2\pi} \sqrt{4-x^2} \mathbb{1}_{[-2,2]}(x)$$



Wigner thm.: $Z_N \rightarrow \sigma(x) dx$ in distribution (weakly), in probability, that is:

$$\forall f \in C_b(\mathbb{R}) \quad (f \text{ bdd. cont. on } \mathbb{R}):$$

$$\mathbb{P} \left(\left| \int f(x) dZ_N(x) - \int f(x) \sigma(x) dx \right| > \varepsilon \right) \xrightarrow{N \rightarrow \infty} 0$$

under the assumptions:

$$1) \mathbb{E} Y_i = \mathbb{E} Z_{i,j} = 0$$

$$2) \mathbb{E} Z_{i,j}^2 = 1$$

$$3) \forall k \geq 2, \max(\mathbb{E}[|Y_i|^k], \mathbb{E}[|Z_{i,j}|^k]) < \infty$$

Idea of method of moments:

$$(\text{Notation: } \int f(x) d\mu(x) =: \langle \mu, f \rangle)$$

$$\langle Z_N, x^k \rangle = \frac{1}{N} \sum_{j=1}^N (X_j^N)^k = \frac{1}{N} \text{Tr}(X_N^k)$$



moments of σ :

prop.:

$$\forall k \geq 0: \quad \langle \sigma, x^k \rangle = \begin{cases} 0 & k \text{ odd} \\ G_{\frac{k}{2}} & k \text{ even} \end{cases}$$

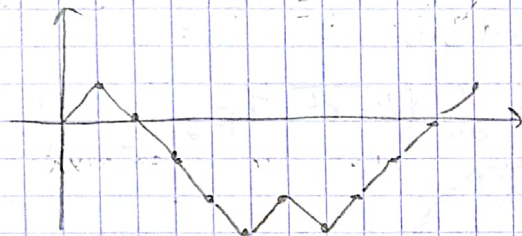
where: (Catalan):

$$G_m = \frac{1}{m+1} \cdot \binom{2m}{m} = \frac{(2m)!}{(m+1)! m!} \quad m \geq 0$$

Short detour on Catalan numbers:

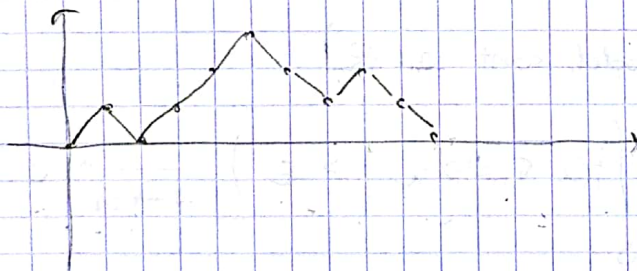
Bernoulli: walk of length $2k$:

$x = (x_0, x_1, \dots, x_{2k})$ where $x_0 = 0, |x_{i+1} - x_i| = 1 \quad \forall i \geq 0$.



Dyck path:

$x_p \geq 0, \forall p$ & $x_{2k} = 0$

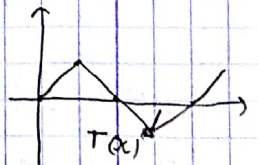


Lemma:

G_k equals the number of Dyck paths of length $2k$.

Proof:

There are $\binom{2k}{k}$ walks of length $2k$ with $x_{2k} = 0$ (exactly k steps go up & k steps go down)



For each walk x that is not a Dyck path, let $T(x)$ be the first index where $x_{T(x)} = -1$. there is a mapping (the reflection mapping) to a walk ending at -2 . This mapping is a bijection. We conclude that the number of Dyck paths of len. $2k$ is:

$$\binom{2k}{k} - \binom{2k}{k-1} = \frac{2k!}{k!k!} - \frac{2k!}{(k-1)!(k+1)!} = C_k$$

□

Lemma:

The generating function of C_k is:

$$f(z) := \sum_{m=0}^{\infty} C_m z^m$$

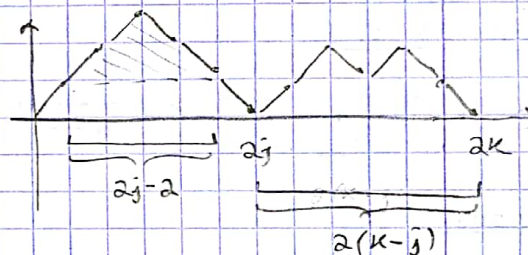
satisfies: $f(z) = \frac{1 - \sqrt{1-4z}}{2z}$, $|z| < \frac{1}{4}$

Proof:

First, $C_m \leq 4^m$ (the dyck paths of length $2k$ are a subset of the bernoulli paths of length $2k$ - there are 2^{2k} of these)

for each Dyck path, by considering the first time $2j$ where it equals 0, get:

$$(*) \quad C_k = \sum_{j=1}^k C_{j-1} C_{k-j}$$



we calculate that:

$$f(z) - 1 = \sum_{k=1}^{\infty} C_k z^k = \sum_{k=1}^{\infty} \sum_{j=1}^k C_{j-1} C_{k-j} z^k$$

$$\& \quad f(z)^2 = \left(\sum_{m=0}^{\infty} C_m z^m \right)^2 = \sum_{k,k'=0}^{\infty} C_k C_{k'} z^{k+k'} = \sum_{q=0}^{\infty} z^q \sum_{j=0}^q C_j C_{q-j}$$

Then: $f(z) - 1 = z(f(z))^2$

This means:

$$f(z) \in \left\{ \frac{1 \pm \sqrt{1-4z}}{2z} \right\} \quad (\text{by solving the quadratic equation})$$

since $f(0) = 1$, $f(z) = \frac{1 - \sqrt{1-4z}}{2z}$

□

Proof of Wigner's thm:

Def.:

$\bar{Z}_n$ is the probability measure $E(Z_n)$. That is, for any measurable $f: \mathbb{R} \rightarrow [0, \infty)$, $\langle \bar{Z}_n, f \rangle := E[\langle Z_n, f \rangle]$ (in particular, for $f = \mathbb{1}_A$, A Borel set).

One checks that $\bar{Z}_n$ is a probability measure.

E.g. σ additivity is the claim that if $A_1, A_2, \dots$ are disjoint

Borel sets, then $\langle \bar{Z}_n, \mathbb{1}_{A_1} + \mathbb{1}_{A_2} + \dots \rangle = \sum_{j=1}^{\infty} \langle \bar{Z}_n, \mathbb{1}_{A_j} \rangle$

by monotone convergence theorem $\rightarrow E[\langle Z_n, \mathbb{1}_{A_j} \rangle]$

$$E[\langle Z_n, \mathbb{1}_{A_1} + \dots \rangle] = E\left[\sum_{j=1}^{\infty} \langle Z_n, \mathbb{1}_{A_j} \rangle\right]$$

Rely on two lemmas for thm:

Lemma 1:

$$\forall k \geq 0. \langle \bar{Z}_n, x^k \rangle \xrightarrow{n \rightarrow \infty} \langle \sigma, x^k \rangle$$

Lemma 2:

$$\text{for any } k \geq 0 \text{ \& } \epsilon > 0. \quad P(|\langle \bar{Z}_n, x^k \rangle - \langle \bar{Z}_n, x^k \rangle| > \epsilon) \xrightarrow{n \rightarrow \infty} 0$$

Now the thm. follows from the lemmas:

Fix $f \in C_b(\mathbb{R})$. Fix $\delta > 0$. Let Q_δ be a poly. satisfying

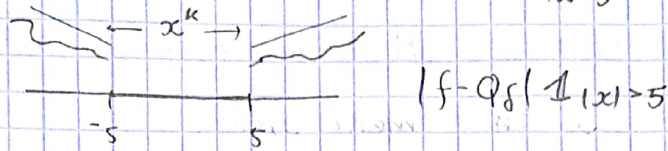
$$|f(x) - Q_\delta(x)| \leq \delta \quad \text{for } -5 \leq x \leq 5$$

$$\begin{aligned} |\langle Z_n, f \rangle - \langle \sigma, f \rangle| &= |\langle Z_n, (f - Q_\delta) \mathbb{1}_{|x| \leq 5} + (f - Q_\delta) \mathbb{1}_{|x| > 5} + Q_\delta \rangle - \\ &\quad - \langle \sigma, (f - Q_\delta) \mathbb{1}_{|x| \leq 5} + (f - Q_\delta) \mathbb{1}_{|x| > 5} + Q_\delta \rangle| \leq \end{aligned}$$

$$\leq \underbrace{2\delta + \langle Z_n, |f - Q_\delta| \mathbb{1}_{|x| \geq 5} \rangle}_{(*)} + \underbrace{|\langle Z_n, Q_\delta \rangle - \langle \sigma, Q_\delta \rangle|}_{(**)}$$

$$\begin{aligned} (**) &\leq \underbrace{|\langle Z_n, Q_\delta \rangle - \langle \bar{Z}_n, Q_\delta \rangle|}_{\text{random}} + \underbrace{|\langle \bar{Z}_n, Q_\delta \rangle - \langle \sigma, Q_\delta \rangle|}_{\text{deterministic}} \xrightarrow{n \rightarrow \infty} 0 \quad (\text{lemma 1}) \\ &\quad \forall \epsilon, P(\cdot > \epsilon) \xrightarrow{n \rightarrow \infty} 0 \quad (\text{lemma 2}) \end{aligned}$$

To control (*) note that $\exists k_0$ s.t. $|f - Q_\delta| \mathbb{1}_{|x| > 5} \leq |x|^k \mathbb{1}_{|x| > 5}, k \geq k_0$



(this is possible because f is bounded)

So that: $(*) = \langle \bar{z}_N, |f - Q_\delta| \mathbb{1}_{|x| > 5} \rangle \leq \langle \bar{z}_N, |x|^k \mathbb{1}_{|x| > 5} \rangle$

$$\forall x, |x|^k \mathbb{1}_{|x| > 5} \leq \frac{x^{2k}}{5^k} \rightarrow \leq \langle \bar{z}_N, x^{2k} \rangle$$

Now we take expectation:

$$\mathbb{E} \langle \bar{z}_N, |f - Q_\delta| \mathbb{1}_{|x| > 5} \rangle \leq \frac{1}{5^k} \langle \bar{z}_N, x^{2k} \rangle \xrightarrow{N \rightarrow \infty} \frac{C_k}{5^k} \leq \left(\frac{4}{5}\right)^k \quad \forall k \geq k_0$$

$$\Rightarrow \limsup_{N \rightarrow \infty} \mathbb{E} \langle \bar{z}_N, |f - Q_\delta| \mathbb{1}_{|x| > 5} \rangle = 0$$

Conclude with Markov's inequality.

Proof of lemma 1:

For $k \geq 1$ (for $k=0$ it is obvious)

$$\langle \bar{z}_N, x^k \rangle = \mathbb{E} \langle \bar{z}_N, x^k \rangle = \frac{1}{N} \mathbb{E} [\text{tr}(X_N^k)]$$

(notice: $(A \cdot B \cdot C)_{ij} = \sum_{m_1, m_2=1}^N A_{im_1} B_{m_1 m_2} C_{m_2 j}$)
 A, B, C - $N \times N$ matrices.

$$\text{Have } \text{Tr}(X_N^k) = \sum_{i_1, i_2, \dots, i_k=1}^N X_N(i_1, i_2) X_N(i_2, i_3) \dots X_N(i_{k-1}, i_k) X_N(i_k, i_1)$$

Denoting: $i = (i_1, \dots, i_k)$ we write

$$T_i^N = X_N(i_1, i_2) \dots X_N(i_k, i_1)$$

Thus, $\langle \bar{z}_N, x^k \rangle = \frac{1}{N} \sum_{i \in \{1, \dots, N\}^k} \mathbb{E}(T_i^N)$

Notation:

Given an alphabet A (in our case $A = \{1, \dots, N\}$),

a letter is an element of A .

a word w is a finite sequence of letters.

a word w is closed if first letter equals last letter

& w_1 is equiv. to w_2 (written $w_1 \sim w_2$) if there is a bijection from A to itself taking w_1 to w_2 .

Example: 1424371 - closed, $\text{supp}(w) = \{1, 2, 3, 4, 7\}$
 $\text{wt}(w) = 5$
 $l(w) = 7$
 2414372

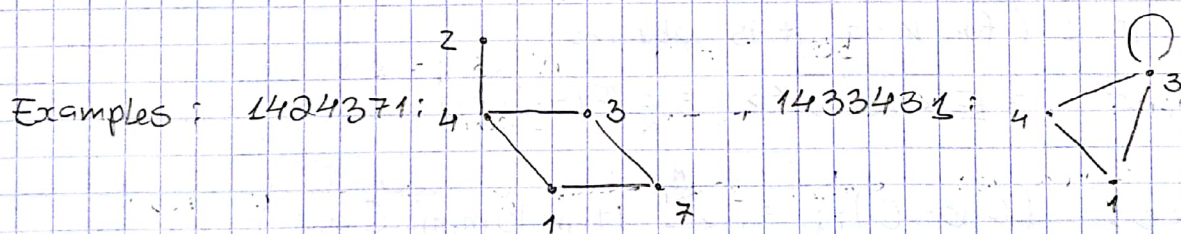
the support of w , $\text{supp}(w)$, is the set of letters in w .

the weight of w is $\text{wt}(w) = |\text{supp}(w)|$

the length of w , denoted $l(w)$, is the length of w as a sequence.

Associated to each word $w = w_1 \dots w_l$ is a graph:

$G_w (V_w, E_w)$ with $V_w = \text{Supp}(w)$, $E_w = \{\{w_i, w_{i+1}\} \mid 1 \leq i \leq l-1\}$



for each edge e , N_e^w is the number of times e was traversed in w .

We also partition E_w into E_w^s , E_w^c

self edges
 $\{v, v\}$

connecting edges
 $\{v, u\}, v \neq u$

With this notation, for any $i \in \{1, \dots, N\}^K$, $w_i = i_1 i_2 \dots i_K$

$$\mathbb{E}(T_i^N) = \frac{1}{N^{K/2}} \prod_{e \in E_{w_i}^c} \mathbb{E}(Z_{4,2}^{N_e^{w_i}}) \prod_{e \in E_{w_i}^s} \mathbb{E}(Y_1^{N_e^{w_i}})$$

Note, if $w_{i_1} \sim w_{i_2}$ then $\mathbb{E} T_{i_1}^N = \mathbb{E} T_{i_2}^N$

write $\omega_{k,t}$ for a set of representatives for equiv. classes of closed words of length $k+1$ & weight t .

Each equiv class of a word in $\omega_{k,t}$ has size:

$$G_{N,t} = N(N-1)(N-2) \dots (N-t+1)$$

$$\text{Thus, } \langle \bar{Z}_N, x^k \rangle = \frac{1}{N} \sum_{i \in \{1, \dots, N\}^k} E T_i^N = \sum_{t=1}^k \frac{G_{N,t}}{N^{k/2+1}} \sum_{\omega \in \omega_{k,t}} \prod_{e \in E_\omega} E(z_{e,1}^N) \prod_{e \in E_\omega} E(z_{e,2}^N)$$

There are k steps in path of ω . To get non-zero expectation, have to pass each edge at least twice. So G_ω has at most $\lfloor \frac{k}{2} \rfloor$ edges.

A connected graph with e edges has $\leq e+1$ vertices, with equality iff graph is a tree.

So G_ω has $\leq \lfloor \frac{k}{2} \rfloor + 1$ vertices. That is $wt(\omega) \leq \lfloor \frac{k}{2} \rfloor + 1$.

Thus,

$$\langle \bar{Z}_N, x^k \rangle = \sum_{t=1}^{\lfloor \frac{k}{2} \rfloor + 1} \frac{G_{N,t}}{N^{k/2+1}} \sum_{\omega \in \omega_{k,t}} |\square| \leq \sum_{t=1}^{\lfloor \frac{k}{2} \rfloor + 1} \frac{G_{N,t}}{N^{k/2+1}} C_k'''$$

$|\square| \leq C_k'$
 $|\omega_{k,t}| \leq C_k''$

the only non-negligible contribution is from $t = \frac{k}{2} + 1$.

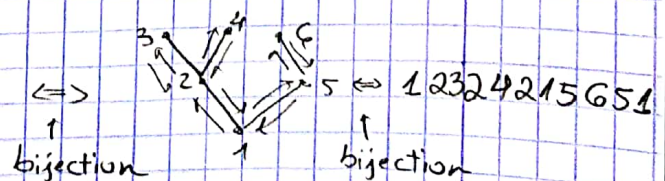
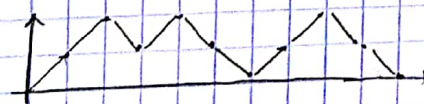
Conclude: $\langle \bar{Z}_N, x^k \rangle \xrightarrow{N \rightarrow \infty} 0$, k odd

for k even, get $\langle \bar{Z}_N, x^k \rangle \xrightarrow{N \rightarrow \infty} \sum_{\omega \in \omega_{k, \frac{k}{2}+1}} |\square| = |\omega_{k, \frac{k}{2}+1}|$

the graph corresponding to $\omega \in \omega_{k, \frac{k}{2}+1}$ is necessarily a tree with edge

traversed exactly twice in path of ω .

Next, $|\omega_{k, \frac{k}{2}+1}|$ equals the number of rooted trees where the children of each vertex are ordered.



More formally, given ω of length $k+1$ & weight $\frac{k}{2} + 1$, define a Dyck path of length k by $x_0 = 0$, $x_i =$ height above root in corresponding tree. This is a bijection.

from this we conclude that $|\omega_{k, \frac{k}{2}+1}| = C_k$.

Altogether, $\langle \bar{Z}_N, x^k \rangle \rightarrow C_k, k \text{ even}$

■

Proof of lemma 2:

Suffices that $\text{Var}(\langle \bar{Z}_N, x^k \rangle) \xrightarrow{N \rightarrow \infty} 0$ (from chebyshev)

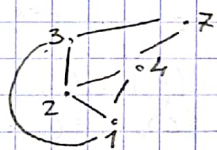
$$\text{Recall, } \langle \bar{Z}_N, x^k \rangle = \frac{1}{N} \sum_{i \in \{1, \dots, N\}^k} T_i^N$$

$$\text{where } T_i^N = X_N(i_1, i_2) \dots X_N(i_k, i_1)$$

$$\text{Hence, } \text{Var}(\langle \bar{Z}_N, x^k \rangle) = \frac{1}{N^2} \sum_{i, i' \in \{1, \dots, N\}^k} \text{Cov}(T_i^N, T_{i'}^N) =$$

$$= \frac{1}{N^2} \sum_{i, i' \in \{1, \dots, N\}^k} \mathbb{E}(T_i^N T_{i'}^N) - \mathbb{E}(T_i^N) \mathbb{E}(T_{i'}^N)$$

$G \cup G'$ - the idea is to draw the two graphs on the same vertices:



$$i = 123241 \quad i' = 14731$$

In order to get Expectance $\neq 0$ we need Each edge traversed ≥ 2 in union of graphs, at least one edge in common to both graphs.