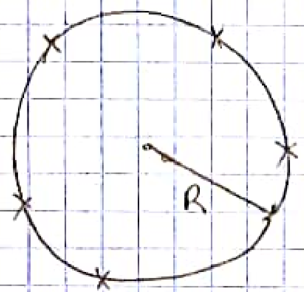
Lecture 9: 16/5/2018

Jarnik's theorem:

An arc of length $\leq cR^{1/3}$ on the circle $x^2+y^2=R^2$ contains at most 2 lattice points.



$R^2 = \Box + \Box$

lattice points on circle $= r_2(R^2) \ll R^{\varepsilon}$ $\quad \forall \varepsilon > 0$

pf:

The proof uses the fact that 3 points on a circle cannot be co linear. So span a triangle of positive area.

If the sides are a, b, c then Herons formula: $abc = 4AR$

$\Longrightarrow \max(a,b,c)^3 \gg AR$

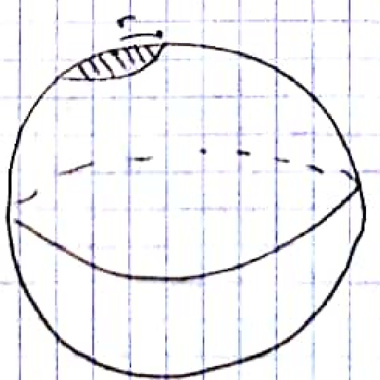
A = area of lattice triangle $\Longrightarrow A \geq \frac{1}{2}$

(since $A = \frac{1}{2}|(u-v)\times(w-v)| \geq \frac{1}{2}$

integer

$\Longrightarrow \max(a,b,c) \gg R^{1/3}$

Exercise: Jarnik in 3-dimensions



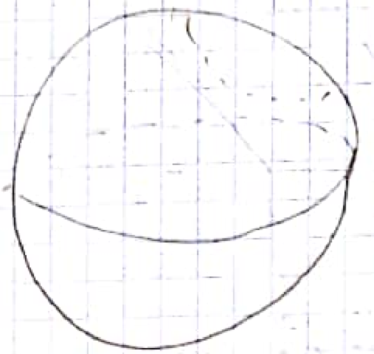
$x^2+y^2+z^2=R^2$

There is no finiteness result. However, $\exists R \longrightarrow \infty$ s.t. $\exists$ cap of size $= R^{\delta}$ contains $\geq \frac{1}{\delta}$ pts.

Jarnak's theorem in 3-dimension:

$\exists\, c>0$ s.t any cap on $x^2+y^2+z^2=R^2$ of diameter $> c\cdot R^{1/4}$ has all lattice in it lying on a plane (co planar)

(This is an exercise)



Conj: (C.lleruelo - Granville) 1990's

$\forall \delta>0\ \exists\, c(\delta)>0$ s.t any arc of diameter $< R^{1-\delta}$ on $x^2+y^2=R^2$ Contains $\leq c(\delta)$ lattice points.

(Jarnik - $1-\delta \leq 1/3$)

Thm: (Cordoba & Cilleruelo)

The statement of the conjecture holds $\forall\ 1-\delta < 1/2$ & gives $c(\delta) \ll \dfrac{1}{\frac{1}{2}-\delta}$

Prop:

If $P_1, P_2, \dots, P_m$ lattice points on $x^2+y^2=R^2$. Then

$$\prod_{1\leq i<j\leq m} |P_i - P_j| > R^{e(m)} \quad \text{where } e(m) = \begin{cases} \frac{m}{2}\left(\frac{m}{2}-1\right) & m \text{ even} \\ \left(\frac{m-1}{2}\right)^2 & m \text{ odd} \end{cases}$$

(we think of $P_i \in \mathbb{Z}[i] \subset \mathbb{C}$ P_i $|P_i - P_j|$ P_j)

Given the Prop. $\Longrightarrow$ $\left(\max |P_i - P_j|\right)^{\binom{m}{2}} > R^{e(m)}$

$\Longrightarrow$ $\max |P_i - P_j| \geq R^{\eta(m)}$, $\eta(m) = \dfrac{e(m)}{\binom{m}{2}}$

Hence an arc of diameter $< R^{\eta(m)}$ Cannot contain m lattice points.

m	3	4	5	6
$e(m)$	1	2	4	6
$\eta(m) = \frac{e(m)}{\binom{m}{2}}$	$1/3$	$\frac{2}{6} = \frac{1}{3}$	$\frac{4}{10} = \frac{2}{5}$	$\frac{6}{15} = \frac{2}{5}$

Jarnik's exponent

When $m \longrightarrow \infty$ $\quad \eta(m) = \frac{e(m)}{\binom{m}{2}} \sim \frac{m^2/4}{m^2/2} = \frac{1}{2}$

<u>Pf of Prop:</u> (Ramana 2007)

Given $p_1, \dots, p_m \in \mathbb{Z}[i]$ (distinct), $p_i \overline{p_i} = R^2$ then $\forall\, 0 \le k \le m-1$

$$(*) \qquad R^{k(k+1)} \prod_{1 \le i < j \le m} (p_j - p_i) = \left(\prod_{i=1}^{m} \overline{p_i}^{k} \right) \det V_{k,m}$$

$$V_{k,m} = \begin{pmatrix} \overline{p_1}^{k} & \overline{p_2}^{k} & \dots & \overline{p_m}^{k} \\ \overline{p_1}^{k-1} & \overline{p_2}^{k-1} & \dots & \overline{p_m}^{k-1} \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & 1 \\ p_1 & p_2 & \dots & p_m \\ \vdots & & & \vdots \\ p_1^{m-1-k} & & \dots & p_m^{m-1-k} \end{pmatrix} \quad m \times m$$

$$(*) \Longrightarrow \prod_{1 \le i < j \le m} |p_i - p_j| = R^{km - k(k+1)} |\det V_{k,m}| \ge$$

$\det V_{k,m} \in \mathbb{Z}[i] \Longrightarrow |\det V| \ge 1$ if $\ne 0$ (Clear because LHS $\ne 0$)

$$\ge R^{k(m-k-1)}$$

Now choose k to maximize $k(m-1-k)$, $0 \le k \le m-1$

($k \sim m/2$)

Scanned by CamScanner

Pf of Ramana's identity: $(*)$

Compute RHS:

$$\rho_1^k \cdot \det\begin{pmatrix} \bar\rho_1^{\,k} & \bar\rho_2^{\,k} & \cdots \\ \bar\rho_1^{\,k-1} & \vdots & \\ \vdots & & \\ \rho_1^{m-k-1} & & \end{pmatrix} = \det\left(\rho_1^k \begin{pmatrix} \text{1st} \\ \text{column} \end{pmatrix},\ \begin{matrix}\text{dont change} \\ \text{other} \\ \text{columns}\end{matrix}\right) =$$

$$= \det\begin{pmatrix} (\rho_1\bar\rho_1)^k & & \\ (\rho_1\bar\rho_1)^{k-1}\rho_1 & & \\ \vdots & \cdots & \\ \rho_1^k & & \\ \rho_1^{k+1} & & \end{pmatrix} = \begin{pmatrix} R^{2k} & \\ R^{2(k-1)}\rho_1 & \\ \vdots & * \\ \rho_1^k & \\ \vdots & \\ \rho_1^{m-1} & \end{pmatrix}$$

Now multiply 2^{nd} column by ρ_2^k, etc.

$$\text{RHS} = \det\begin{pmatrix} R^{2k} & R^{2k} & \cdots & R^{2k} \\ R^{2(k-1)}\rho_1 & & & R^{2(k-1)}\rho_m \\ \vdots & & & \\ \rho_1^k & \rho_2^k & & \rho_m^k \\ \vdots & & & \\ \rho_1^{m-1} & & \cdots & \rho_m^{m-1} \end{pmatrix} = R^{2k} R^{2(k-1)} \cdots R^2 \det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \rho_1 & \rho_2 & \cdots & \rho_m \\ \vdots & & & \vdots \\ \rho_1^{k-1} & & \cdots & \rho_m^{k-1} \\ \vdots & & & \vdots \\ \rho_1^{m-1} & & \cdots & \rho_m^{m-1} \end{pmatrix}$$

$$= R^{k(k+1)} \det(\text{Vandermonde}) = R^{k(k+1)} \prod_{1\le i<j\le m} (\rho_j - \rho_i)$$

$\blacksquare$

Examining the exponents, will get that for $\delta > 0$

$$\#\begin{pmatrix} \text{lattice pts on arc of} \\ \text{size } R^{1/2-\delta} \end{pmatrix} \ll_\delta$$

Question: Can improve exponent $1/2$?

"Evidence" for conjecture:

1) The average distance between lattice points on $x^2+y^2=R^2$ is

$$\frac{\text{length of circumfrance}}{\#\text{ pts}} = \frac{2\pi R}{r_2(R^2)} \gg R^{1-\varepsilon} \quad \forall \varepsilon>0$$

proved $r_2(n) \ll n^{\varepsilon} \ \forall \varepsilon>0$.

So Average nearest neighbour distance $\gg R^{1-\varepsilon}$

So maybe worst case is not far from average.

2) Thm.:

For "almost all" R (s.t $R^2 = \square + \square$) the conj. is true:

$$\forall \varepsilon>0 \quad \min_{i<j} |P_i - P_j| > R^{1-\varepsilon}$$

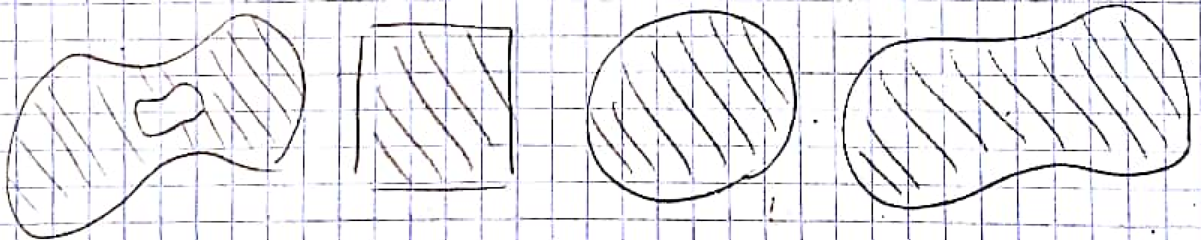
In the sense that $\#\{\text{exceptions } \underset{\square+\square}{E=R^2} \leq X\} \ll X^{1-\varepsilon/3}$

Recall $\#\{E=\square+\square \leq X\} \sim K\cdot x/\sqrt{\log x}$

Connection with eigenfunctions of Laplacian

The problem: "Can we hear the shape of a drum"?

"drum" = nice planar domain $\Omega \subseteq \mathbb{R}^2$



"nice" "=" compact, piecewise smooth boundary, connected.

"hear" we can hear the eigenvalues of the Dirichlet Laplacian.

$$-\Delta f = Ef \quad , \quad f \in C^\infty(\Omega), \quad f|_{\partial\Omega} = 0, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Can hear the E's.

So ask: Given the eigenvalues $\{E_n\}_{n=1}^\infty$, Can I recover the domain Ω (up to translation & rotation)

Example:

dimension 1 :  ; $\Omega = [0, A]$ interval

$\Delta = \frac{d^2}{dx^2}$: $-f'' = Ef$, $f(0) = 0 = f(A)$, $f \in C^\infty([0,A])$

Solutions : $f_n(x) = \sqrt{2} \sin(\pi n \frac{x}{A})$, $n = 1, 2, 3, \ldots$

$E_n = \pi^2 (n/A)^2 = (\frac{\pi}{A})^2 n^2$

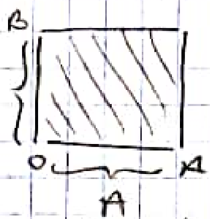
Recover A from E_n : $\# \{ E_n \leq X \}$

$(\frac{\pi}{A} \cdot n)^2 \leq x \Rightarrow 1 \leq n \leq \frac{\sqrt{x}}{\pi / A} = \frac{A}{\pi} \sqrt{x}$

So $\# \{ E_n \leq X \} = \frac{A}{\pi} x^{1/2} + O(1)$,

So recovered $[0, A]$ from spectrum $x \to \infty$.

Example 2:

2-dim. Ω =  $= [0, A] \times [0, B] = R$

$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ take $f_{m,n}(x,y) = \frac{2}{\sqrt{AB}} \cdot \sin\left(\frac{\pi m x}{A}\right) \sin\left(\frac{\pi n y}{B}\right)$, $m, n \geq 1$

the $-\Delta f_{m,n} = \pi^2 \left((\frac{m}{A})^2 + (\frac{n}{B})^2 \right) f$; $f_{m,n}|_{\partial R} = 0$

Fact

$\{ f_{m,n} \}$ is an orthonormal basis of L^2 (Rectangle)

Spectrum $E_{m,n} = \pi^2 \left((\frac{m}{A})^2 + (\frac{n}{B})^2 \right)$, $m, n \geq 1$

What can we recover about R from $\mathrm{Spec}(\Delta_R)$?

Can recover $\mathrm{area}(R) = A \cdot B$.

Spectral Staircase. $N(X) = \# \{ E_{m,n} \leq X \}$

Claim:

$$N(X) \underset{X \to \infty}{\sim} \frac{\mathrm{area}(R)}{4\pi} \cdot X$$

So know area (R)

Proof:

$$N(x) = \#\{(m,n),\ m,n \geq 1 \mid \pi^2\left(\left(\tfrac{m}{A}\right)^2 + \left(\tfrac{n}{B}\right)^2\right) \leq x\} = \mathbb{Z}^2 \cap \tfrac{1}{4}\text{ ellipse}$$

$$\Longrightarrow \text{know } N(x) \sim \tfrac{1}{4}\operatorname{area}(\text{ellipse}) = \frac{x}{\pi^2} \cdot \frac{1}{4}\operatorname{area}\left[\left(\tfrac{x}{A}\right)^2 + \left(\tfrac{y}{B}\right)^2 \leq 1\right] =$$

$$= \frac{AB}{4\pi} \cdot x = \frac{\operatorname{area}(\Omega)}{4\pi} x$$

Weyl's law (1911):

$\Omega \subset \mathbb{R}^2$ nice planar domain, $\operatorname{Spec}(\Delta) = \{E_1, E_2, \dots E_n, \dots\}$,

eigenfunctions give ONB (orthonormal basis) for $L^2(\Omega)$.

$$N(x) := \#\{E_n \leq x\} \underset{x \to \infty}{\sim} \frac{\operatorname{area}(\Omega)}{4\pi} \cdot x$$

So can "hear" the area of a drum!

Sometime during the 1990's Gordon Webb Wolpert:

Found 2 non-isometric domains with same spectrum.

Milnor 1950's: $\exists$ two nonisometric flat tori $\mathbb{R}^{16}/L$

which are isospectral

$h = 2$

Can hear: 1) the length of $\partial\Omega$

2) # of holes $h(\Omega)$

Use the "trace of heat kernel"

$$\Theta(t) := \sum_{n=1}^{\infty} e^{-tE_n}, \quad t > 0 \quad \text{(use Weyl to check convergence)}$$

$$\Theta(t) \underset{t \downarrow 0}{\sim} c_1 \cdot \frac{\operatorname{area}(\Omega)}{t} \leftarrow \text{exercise using Weyl \& summation by parts.}$$

In fact $$\sim c_1 \frac{\operatorname{area}(\Omega)}{t} - c_2 \frac{\operatorname{length}(\partial\Omega)}{\sqrt{t}} - \frac{h(\Omega) - 1}{6} + o(1)$$

$$c_1 = \frac{1}{4\pi}, \quad c_2 = \frac{1}{8\sqrt{\pi}}$$

Exercise: For 1-Dim case $\Omega = [0, A]$, $E_n = \underbrace{\left(\frac{\pi^2}{A^2}\right)}_{\overset{\shortparallel}{1}} n^2$

$$\omega(t) = \sum_{m=1}^{\infty} e^{-tm^2}, \quad \omega \underset{t \to +\infty}{\sim} e^{-t}, \quad \omega(t) = \frac{\Theta(t) - 1}{2}, \quad \Theta(t) = \sum_{m=-\infty}^{\infty} e^{-tm^2}$$

Thm: $\Theta(t) \underset{t \downarrow 0}{\sim} \frac{1}{\sqrt{t}}$ $\quad \boxed{\Theta\left(\tfrac{1}{t}\right) = \sqrt{t}\,\Theta(t)}$

For Ω = unit disk, the eigenfunction:

$$f_{n,k,\pm}(r,\theta) = J_n(j_{n,k} r) \begin{cases} \cos(n\theta) \\ \sin(n\theta) \end{cases}$$

$n = 1, 2, \ldots$

$k = 1, 2, 3, \ldots$

J_n - n-th Bessel function $j_{n,1}, j_{n,2}, \ldots$ = zeros of $J_n(x)$

$$-\Delta f_{n,k,\pm} = j^2_{n,k} f_{n,k,\pm}$$

← eigenvalues

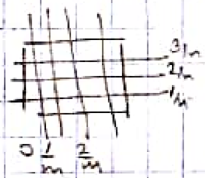
Nodal line $= \{ \vec{x} \in \Omega \mid f(x) = 0 \} =: Z_f$

$-\Delta f = Ef$ eigenfunction

Question:

How does Z_f change with E eigenvalues?

Examples:

Ω =  $f_{m,n}(x,y) = \sin(\pi m x)\sin(\pi n y)$

Question:

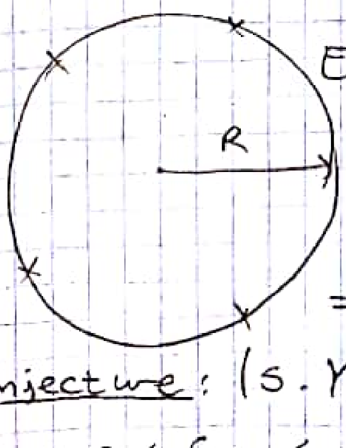
What can we say about length (Z_f)?

Example:

Unit Square $E = m^2 + n^2$

$Z(f_{m,n})$ = union of $m + O(1) + n + O(1)$ unit segments.

length $(Z(f_{m,n})) = m + n + O(1)$



$E = m^2 + n^2$

length

$\frac{1}{100}\sqrt{E} \leq m + n \leq \sqrt{2E}$

$(1 \cdot m + 1 \cdot n \leq \sqrt{1^2+1^2}\sqrt{m^2+n^2})$

$\Longrightarrow$ In this example $c\sqrt{E} \leq \text{length}(f_{m,n}) \leq C\sqrt{E}$

Conjecture: (S. Yau - 1984)

$$0 < c_\Omega \leq \frac{\text{length}(Z_f)}{\sqrt{E}} \leq C_\Omega < \infty$$

Courant's nodal line theorem (1923): (Courant Hilbert)

Let ν_n = # of "nodal domains" of n-th eigenfunction

" connected component of $\Omega \setminus Z_f$

Then $\nu_n \leq n$.

Pleijel (1956):

$$\frac{\nu_n}{n} \leq 0.691 = \frac{4}{j_{0,1}^2}$$