

The Arc-ComplexDef:

A simplicial complex X on vertex set Σ , is a collection of finite subsets of Σ s.t if $E \subseteq F \in X \Rightarrow E \in X$

Whenever $E \subseteq F \in X$, E is called a face of F .

The geometric realization of X consists of the union of $\{\Delta_E \mid E \in X\}$,
 Δ_E is a ^(closed) Euclidean simplex of dimension $|E|-1$.



Whenever $E \subseteq F$ we identify Δ_E with the corresponding face of Δ_F .

The star of $E \in X$ is the subcomplex consisting of $\{\Delta_F \mid E \subseteq F\}$

The link of $E \in X$ is the subcomplex consisting of $\{\Delta_A \mid A \cap E = \emptyset, A \cup E \in X\} = \{\Delta_A \mid A = F \setminus E, F \in \text{Star}(E)\}$

Thm (Haver '85):

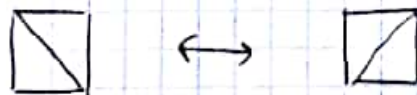
$A(S, V)$ is either: (i) empty

or: (ii) homeomorphic to a sphere if $S = \text{disc}$, $V \subseteq \partial S$
 $|V| \geq 4$

or: (iii) Contractible otherwise

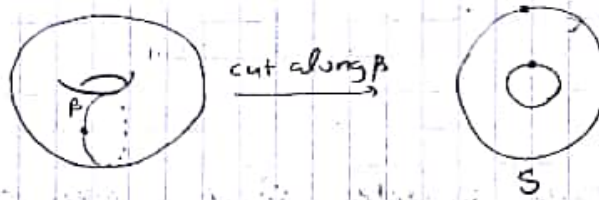
Cor:

Every two triangulations of (S, V) are connected by a finite sequence of moves

pf of corollary:Lemma:

If E is a simplex in $A(S, V)$ of $\text{codim} \geq 2$ then $\text{link}(E)$ is connected.

For example:



then $\text{link}(\beta) = A(S, V) \approx \text{---}$

pf of cor.:

By the thm. $\exists$ path in $A(S, V)$ between any two top-dim simplices. We need to show there is a path passing only through top-dim simplices & co-dim-1 simplices.

We take the best path. If the minimal simplex has $\text{codim} \geq 2$, we bypass it & use its star relying on the lemma.

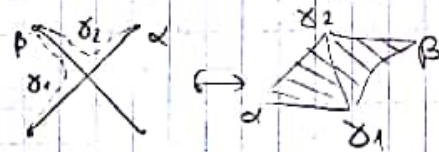
Proof of thm.:

Let $[\alpha], [\beta]$ vertices in $A(S, V)$

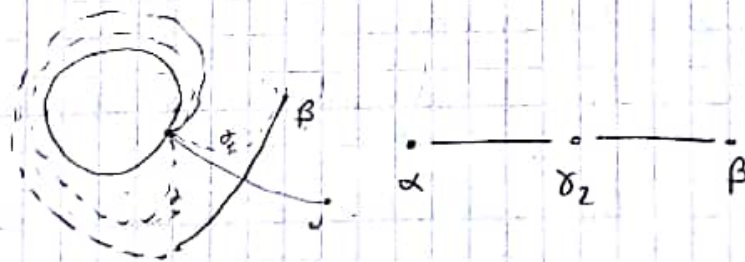
case 0: $|\alpha \cap \beta| = 0$ $\alpha \longrightarrow \beta$

case 1: $|\alpha \cap \beta| = 1$

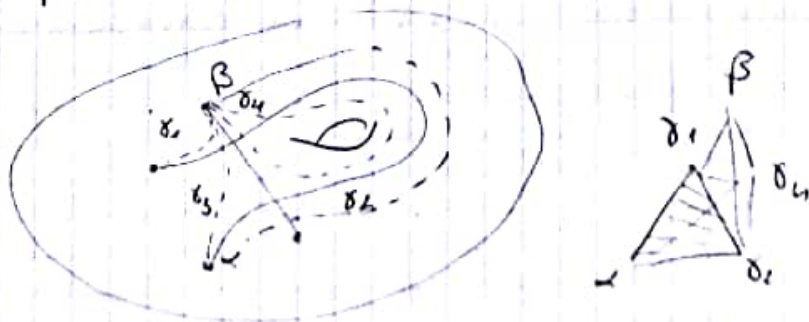
(problem, δ_1 or δ_2 can be isotopic to the boundary therefore we make the following assumption)



Assume: Exactly one vertex of V in each Component of ∂S



case 2: $|\alpha \cap \beta| = 2$



The proof of the thm will follow from:

Prop. 1:

If there is exactly one vertex of V in every component of $\mathcal{AS}$ & $A(S, V) \neq \emptyset$ then $A(S, V)$ is contractible.

Prop. 2:

If V' is obtained from V by adding v' to some component of $\mathcal{AS}$ already containing a pt, $v \in V$; then $A(S, V')$ is homeomorphic to the suspension $\sum A(S, V)$ of $A(S, V)$

$$\sum X = I \times X / \begin{matrix} (x_1, 0) \sim (x_2, 0) \\ (x_1, 1) \sim (x_2, 1) \\ \forall x_1, x_2 \in X \end{matrix}$$