

Lecture 5:The circle problem

$$N(R) := \# \mathbb{Z}^2 \cap \text{Ball}(0, R)$$

Showed: $N(R) = \pi R^2 + O(R)$
 'length of bdy.

Problem Good upper bound for

$$|p(R)| := |N(R) - \pi R^2|$$

Conj

$$p(R) = O(R^{1/2 + o(1)})$$

Goals:

1) $p(R) = O(R^{2/3})$ * 1905 Sierpinski, Voronoi Van der Corput 1922

2) $p(R) = \Omega(R^{1/2})$ i.e. $\exists c > 0 \exists$ arbitrarily large

R 's s.t. $|p(R)| > cR^{1/2}$

Oscillatory Integrals

$$I(\lambda) := \int_{-\infty}^{\infty} A(x) e^{i\lambda\phi(x)} dx$$

$A(x) \in$ Compactly supported — "Amplitude"

$\phi(x)$ real valued smooth — "phase"

(Goal: The Fourier transform of the unit ball
 $\int_{x^2+y^2 \leq 1} e^{2\pi i(ax+by)} dx dy, (a,b)' \rightarrow \infty$)

Trivial Bound:

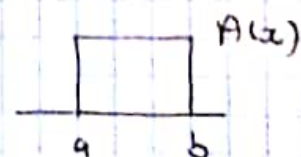
$$|I(\lambda)| \leq \int |A(x)| dx$$

Want: $I(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$

Example:

$$A(x) = 1_{[a,b]}$$

$$\int_a^b e^{i\lambda\phi(x)} dx$$



Example:

$$J_0(\lambda) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\lambda \overbrace{\sin(t)}^{\phi(t)}} dt \quad - \text{Bessels function}$$

$$J_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{z}{2}\right)^m$$

Thm:

$$J_0(\lambda) \ll \frac{1}{\sqrt{\lambda}} \quad \lambda \rightarrow \infty$$

From the summation representation we get $J_0(z) \sim_{z \rightarrow 0} 1$

Principle of non-stationary phase

If $A(x) \in C_c^\infty(\mathbb{R})$ is smooth, compactly supported &

$\phi' \neq 0$ on support(A)

then $I(\lambda) \ll \frac{1}{\lambda^N} \quad \forall N \geq 1$

Proof:

Integrate by parts.

$$\begin{aligned} \int A(x) e^{i\lambda \phi(x)} dx &= \int \frac{A(x)}{i\lambda \phi'(x)} \underbrace{\frac{d}{dx}(e^{i\lambda \phi(x)})}_{\phi'(x)} dx = \\ &= \frac{A(x)}{i\lambda \phi'(x)} e^{i\lambda \phi(x)} \Big|_{-\infty}^{\infty} - \frac{1}{i\lambda} \int \underbrace{\frac{d}{dx} \left\{ \frac{A(x)}{\phi'(x)} \right\}}_{A_1(x)} e^{i\lambda \phi(x)} dx \end{aligned}$$

" 0

$$|I(\lambda)| \leq \frac{1}{\lambda} \left| \int A_1(x) e^{i\lambda \phi(x)} dx \right| \leq \frac{1}{\lambda} \int |A_1(x)| dx \ll \frac{1}{\lambda}$$

Now iterate N times.

□

Van der Corput's Lemma 1:

Assume ϕ real, smooth, $\phi' \neq 0$ on $[a, b]$ & ϕ' monotonic.

$$I(\lambda) = \int_a^b e^{i\lambda \phi(x)} dx, \quad \text{Then}$$

$$|I(\lambda)| \leq \frac{4}{\min_{x \in [a, b]} |\phi'(x)|} \cdot \frac{1}{\lambda}$$

Proof:

Integrate by parts.

$$\begin{aligned} \int_a^b e^{i\lambda\phi(x)} dx &= \int_a^b \frac{i\lambda\phi'(x) e^{i\lambda\phi(x)}}{i\lambda\phi'(x)} dx = \int_a^b \frac{1}{i\lambda\phi'(x)} \cdot \frac{d}{dx} \{ e^{i\lambda\phi(x)} \} dx = \\ &= \frac{1}{i\lambda\phi'(b)} e^{i\lambda\phi(b)} - \int_a^b \frac{d}{dx} \left\{ \frac{1}{i\lambda\phi'(x)} \right\} e^{i\lambda\phi(x)} dx \end{aligned}$$

Boundary term

$$\left| \frac{1}{i\lambda} \left\{ \frac{e^{i\lambda\phi(b)}}{\phi'(b)} - \frac{e^{i\lambda\phi(a)}}{\phi'(a)} \right\} \right| \leq \frac{1}{\lambda} \cdot \frac{2}{\min_{[a,b]} |\phi'|}$$

$$\left| \int_a^b \frac{d}{dx} \left\{ \frac{1}{i\lambda\phi'(x)} \right\} e^{i\lambda\phi(x)} dx \right| \leq \frac{1}{\lambda} \int_a^b \left| \frac{d}{dx} \left\{ \frac{1}{\phi'(x)} \right\} \right| \cdot 1 dx$$

ϕ' monotonic $\Rightarrow \frac{1}{\phi'}$ monotone $\Rightarrow \frac{d}{dx} \left\{ \frac{1}{\phi'} \right\}$ has a fixed sign in $[a,b] \Rightarrow \left| \int x \right| = \left| \int x \right|$

$$\int_a^b \left| \frac{d}{dx} \left\{ \frac{1}{\phi'} \right\} \right| dx = \left| \int_a^b \frac{d}{dx} \left\{ \frac{1}{\phi'} \right\} dx \right| = \left| \frac{1}{\phi'(b)} - \frac{1}{\phi'(a)} \right| \leq \frac{2}{\min_{[a,b]} |\phi'|}$$

$$\Rightarrow |I(\lambda)| \leq \frac{L}{\min_{[a,b]} |\phi'|} \cdot \frac{1}{\lambda}, \quad \lambda > 0.$$

Van der Corput's Lemma 2:

Assume ϕ real, smooth & $\phi'' \neq 0$ on $[a,b]$

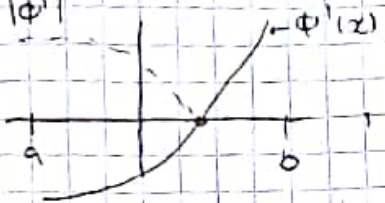
then

$$I(\lambda) = \int_a^b e^{i\lambda\phi(x)} dx \leq \frac{8}{(\min_{[a,b]} |\phi''|)^{1/2}} \cdot \frac{1}{\sqrt{\lambda}}$$

More generally, if $k \geq 2$ & $\phi^{(k)}(x) \neq 0$ on $[a,b]$ then:

$$|I(\lambda)| \leq \frac{C_k}{(\min_{[a,b]} |\phi^{(k)}|)^{1/k}} \cdot \frac{1}{\lambda^{1/k}}$$

$C_k > 0$ absolute constant.

Proof of VDC2: $|\phi'|$  let c be such that

$|\phi'(c)|$ is minimal.

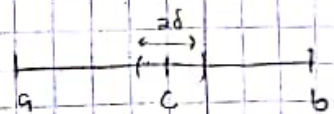
Two cases: $a < c < b$ or $c = a, b$

Assume $a < c < b$

Rescale: replace ϕ by $\phi_1 := \frac{\phi}{\min |\phi''|}$; now $|\phi_1''| \geq 1$

$$\int e^{i\lambda\phi} = \int e^{i(\min |\phi''| \cdot \lambda) \frac{\phi}{\min |\phi''|}} = \tilde{I}(\min |\phi''| \cdot \lambda)$$

Take $\delta > 0$ small; then outside of $(c-\delta, c+\delta)$



we have $|\phi_1'(t)| \geq \delta$

$$\phi_1'(t) = \phi_1(t) - \phi_1(c) = \int_c^t \phi_1''(x) dx$$

$$(t > c+\delta) \quad \phi_1'' \geq 1 \Rightarrow \phi_1'(t) \geq \int_c^t 1 = t - c > \delta$$

use VDC1 on $[a, c-\delta]$ & $[c+\delta, b]$

$$\left| \int_{c-\delta}^{c+\delta} e^{i\lambda\phi_1(x)} dx \right| \leq \frac{4}{\delta} \cdot \frac{1}{\lambda} \quad \left| \int_a^{c-\delta} e^{i\lambda\phi_1(x)} dx \right| \leq \frac{4}{\delta\lambda}$$

$$\left| \int_{c-\delta}^{c+\delta} e^{i\lambda\phi_1(x)} dx \right| \leq \int_{c-\delta}^{c+\delta} 1 = 2\delta$$

trivial

$$|I(\lambda)| = \left| \int_a^b \right| = \left| \int_a^{c-\delta} + \int_{c-\delta}^{c+\delta} + \int_{c+\delta}^b \right| \leq \frac{4}{\delta\lambda} + 2\delta + \frac{4}{\delta\lambda}$$

choose δ to minimize RHS $\Rightarrow I(\lambda) \leq \frac{8}{\sqrt{\lambda}}$ & this is for ϕ_1

Corollary

$A \in C^1[a, b]$, ϕ real, smooth, $\phi^{(k)} \neq 0$ on $[a, b]$ (if $k=1$, ϕ' is monotone)

Then

$$\left| \int_a^b A(x) e^{i\lambda\phi(x)} dx \right| \leq \frac{C_k}{(\min_{[a,b]} |\phi^{(k)}|)^{1/k}} (|A(b)| + \int_a^b |A'(x)| dx) \cdot \frac{1}{\lambda^{1/k}}$$

Proof:

Integrate by parts.

$$\int_a^b A(x) e^{i\lambda\phi(x)} dx = \int_a^b A(x) \frac{d}{dx} J_\lambda dx = A(x) J_\lambda(x) \Big|_a^b - \int_a^b A'(x) J_\lambda(x) dx$$

$$J_\lambda(x) := \int_a^x e^{i\lambda\phi(x)} dx$$

$$\leq A(b) \int_a^b e^{i\lambda\phi(x)} dx + \int_a^b |A'(x)| \frac{C_k}{\min_{[a,b]} \phi'(x)} \frac{1}{\lambda^{1/k}} dx$$

$$\forall \epsilon > 0 \rightarrow \lambda \gg \frac{1}{\epsilon^{1/k}}$$

Main Application - the Fourier transform of the unit disk.

$$\hat{\chi}(\xi) := \int_{|x| \leq 1} e^{ix \cdot \xi} dx, \quad \xi \in \mathbb{R}^2$$

$$= \int_{\mathbb{R}^2} \mathbb{1}_{B(0,1)}(x) e^{ix \cdot \xi} dx$$

Thm:

$$|\hat{\chi}(\xi)| \ll \frac{1}{|\xi|^{3/2}} \quad |\xi| \geq 1$$

Baby example: 1 Dimensional case = Fourier transform of unit interval.

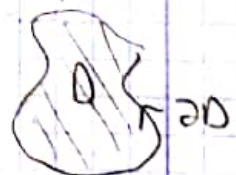
$$\int_{|x| \leq 1, x \in \mathbb{R}} e^{ix\xi} dx = \int_{-1}^1 e^{ix\xi} dx \stackrel{\xi \neq 0}{=} \frac{e^{ix\xi}}{i\xi} \Big|_{-1}^1 = \frac{e^{i\xi} - e^{-i\xi}}{i\xi} = \frac{2\sin \xi}{\xi}$$

$$\text{so get } |\hat{\chi}| \ll \frac{1}{|\xi|}$$

Back to 2 dim reduce to a 1-dim integral via

Green's Theorem

$$\int_{\partial D} A dx + B dy = \iint_D \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) dx dy$$



For us $D = B(0,1)$ $\partial D = S^1$.

$$\text{Want: } \frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} = e^{i(x,y) \cdot \xi} \quad \xi = (m,n)$$

$$A = e^{i\xi(x,y)} \frac{in}{|\xi|^2}, \quad B = -e^{i\xi(x,y)} \frac{im}{|\xi|^2} = -\frac{im}{m^2+n^2} e^{i(mx+ny)}$$

$$\left. \begin{aligned} \frac{\partial B}{\partial x} &= \frac{in(-im)}{m^2+n^2} e^{i(mx+ny)} \\ \frac{\partial A}{\partial y} &= \frac{in \cdot in}{m^2+n^2} e^{i(mx+ny)} \end{aligned} \right\} \Rightarrow \frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} = e^{i(mx+ny)}$$

$$\hat{X}(\xi) := \int_{|x| \leq 1} e^{ix \cdot \xi} dx = \int_{S^1} \frac{in}{m^2+n^2} e^{i(mx+ny)} dx - \frac{im}{m^2+n^2} e^{i(mx+ny)} dy$$

Parametrize $\gamma(t) = (\cos t, \sin t) = \text{unit speed parameterization}$

$$\begin{aligned} \hat{X}(\xi) &= \int_0^{2\pi} \left\langle \frac{in}{m^2+n^2}, \frac{-im}{m^2+n^2} \right\rangle, \gamma(t) \cdot e^{i\langle \xi, \gamma(t) \rangle} dt = \\ &= \frac{i}{|\xi|} \int_0^{2\pi} \left\langle \frac{\xi^\perp}{|\xi|}, \dot{\gamma}(t) \right\rangle e^{i\lambda \langle \frac{\xi}{|\xi|}, \gamma(t) \rangle} dt \end{aligned}$$

$$\xi = (m, n), \quad \xi^\perp = (n, -m), \quad \xi \cdot \xi^\perp = 0$$

$$\hat{X}(\xi) = \frac{i}{|\xi|} \int_0^{2\pi} A_\xi(t) e^{i\lambda \Phi_\xi(t)} dt, \quad \lambda = |\xi| \rightarrow \infty$$

$$A_\xi(t) = \left\langle \frac{\xi^\perp}{|\xi|}, \dot{\gamma}(t) \right\rangle, \quad \Phi_\xi(t) = \left\langle \frac{\xi}{|\xi|}, \gamma(t) \right\rangle$$

Critical points of the phase function Φ_ξ :

$$\Phi'_\xi(t) = \left\langle \frac{\xi}{|\xi|}, \dot{\gamma}(t) \right\rangle \stackrel{?}{=} 0 \Leftrightarrow \dot{\gamma}(t) \perp \xi$$

There will be 2 critical pts, $t_0, t_0 + \pi$.

$$\gamma(t) = (\cos t, \sin t), \quad \dot{\gamma}(t) = (-\sin t, \cos t), \quad \ddot{\gamma}(t) = (-\cos t, -\sin t) = -\gamma(t)$$

$$\Phi''_\xi(t) = \left\langle \frac{\xi}{|\xi|}, \ddot{\gamma}(t) \right\rangle$$

$$\Phi''(t) = 0 \Leftrightarrow \ddot{\gamma}(t) \perp \xi$$

Note We can not have both $\Phi'(t) = 0 = \Phi''(t) \Leftrightarrow \dot{\gamma}(t) \perp \xi, \ddot{\gamma}(t) \perp \xi$

$\Rightarrow \ddot{\gamma}, \dot{\gamma}$ are parallel but $\dot{\gamma} \perp \ddot{\gamma}$. Real reason: γ is a unit speed param

$$\dot{\gamma}(t) \cdot \dot{\gamma}(t) = 1 \Rightarrow 0 = \frac{d}{dt} 1 = 2\dot{\gamma} \cdot \ddot{\gamma}$$

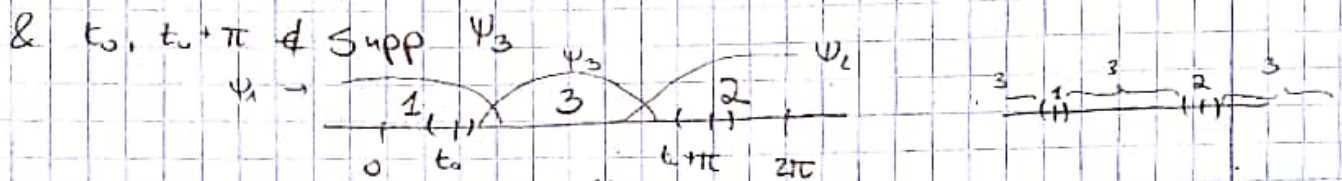
Vdc:

$$\left| \int_a^b A(t) e^{i\lambda \phi(t)} dt \right| \leq \left\{ |A(b)| \cdot \int_a^b |A'(t)| dt \right\} \frac{C_k}{(\min_{[a,b]} |\phi'(t)|)^{1/k}} \frac{1}{\lambda^{1/k}}$$

if $\phi'(t) \neq 0$ on $[a,b]$ (& monotone if $k=1$)

Use a smooth partition of unity: $[0, 2\pi] = \psi_1 + \psi_2 + \psi_3$

When ψ_j smooth, $\text{Supp}(\psi_1) \ni t_0$, $\text{Supp}(\psi_2) \ni t_0 + \pi$



$$\int A_{\xi}(t) e^{i\lambda \phi_{\xi}(t)} dt = \sum_{j=1}^3 \int_0^{2\pi} \underbrace{A_{\xi}(t) \psi_j(t)}_{A_j(t)} e^{i\lambda \phi_{\xi}(t)} dt$$

For $j=1,2$ - use vdc 2, $j=3$ use vdc 1

Let's bound I_1, I_2 (each contains a single crit. pt)
us vdc 2

$$\phi_{\xi}''(t) = \left\langle \frac{\xi}{|\xi|}, \ddot{\gamma}(t) \right\rangle$$

$$\text{Know } \ddot{\gamma}(t) \perp \dot{\gamma}(t), \quad \ddot{\gamma}(t_0) \perp \xi \Rightarrow \ddot{\gamma}(t_0) = \pm \frac{\xi^{\perp}}{|\xi|}$$

$$\Rightarrow \ddot{\gamma}(t_0) = \pm \frac{\xi}{|\xi|}$$

Near t_0 , $\ddot{\gamma}(t)$ is close to $\xi/|\xi|$ (WZOG) so,

$$\phi_{\xi}''(t) = \left\langle \frac{\xi}{|\xi|}, \ddot{\gamma}(t) \right\rangle \approx \left\langle \frac{\xi}{|\xi|}, \frac{\xi}{|\xi|} \right\rangle = 1$$

i.e. $\phi_{\xi}''(t) = 1$ & is $\geq \frac{1}{2}$ say in $(t_0 - \epsilon, t_0 + \epsilon)$

$$\Rightarrow |\phi''(t)| \geq 1/2 \text{ if } \text{Supp } \psi_1 = [t_0 - \epsilon, t_0 + \epsilon]$$

$$|A_1(t)| = \psi_1(t) \left| \left\langle \frac{\xi^{\perp}}{|\xi|}, \ddot{\gamma}(t) \right\rangle \right| \leq 2, \text{ likewise } A'(t) \leq 0$$

$$\leq 2 \left| \frac{\xi^{\perp}}{|\xi|} \cdot \ddot{\gamma}(t) \right|$$

$$\Rightarrow I_1 \ll \frac{1}{|\xi|^{1/2}}$$

Exercise: $I_3 \ll \frac{1}{|\xi|}$

$$\Rightarrow \hat{X}(\xi) \ll \frac{1}{|\xi|^{3/2}}$$

General Thm:

$\subset \mathbb{R}^2$ planar
Let D be a compact domain smooth bdy ∂D , s.t

∂D has nowhere zero curvature, then the Fourier transform of D :

$$\left| \int_D e^{ix\xi} dx \right| \ll \frac{1}{|\xi|^{3/2}}$$

If $\gamma: [0, 1] \rightarrow \partial D$ is a unit parametrization
 $|\dot{\gamma}(t)| \equiv 1$, then the curvature is $K(t) = |\ddot{\gamma}(t)|$

The formula of stationary phase

$$I(\lambda) = \int A(x) e^{i\lambda\phi(x)} dx$$

Assume $A \in C_c^\infty(\mathbb{R})$ smooth, $\phi \in C^\infty(\mathbb{R})$ real valued

Suppose ϕ has a single critical pt. $x_0 \in \text{Supp}(A)$, $\phi'(x_0) = 0$,

which is non-degenerate: $\phi''(x_0) \neq 0$.

Then

$$I(\lambda) \underset{\lambda \rightarrow \infty}{\sim} e^{\frac{i\pi}{4} \text{sign } \phi''(x_0)} \cdot A(x_0) \sqrt{\frac{2\pi}{|\phi''(x_0)|}} \cdot \frac{e^{i\lambda\phi(x_0)}}{\sqrt{\lambda}}$$