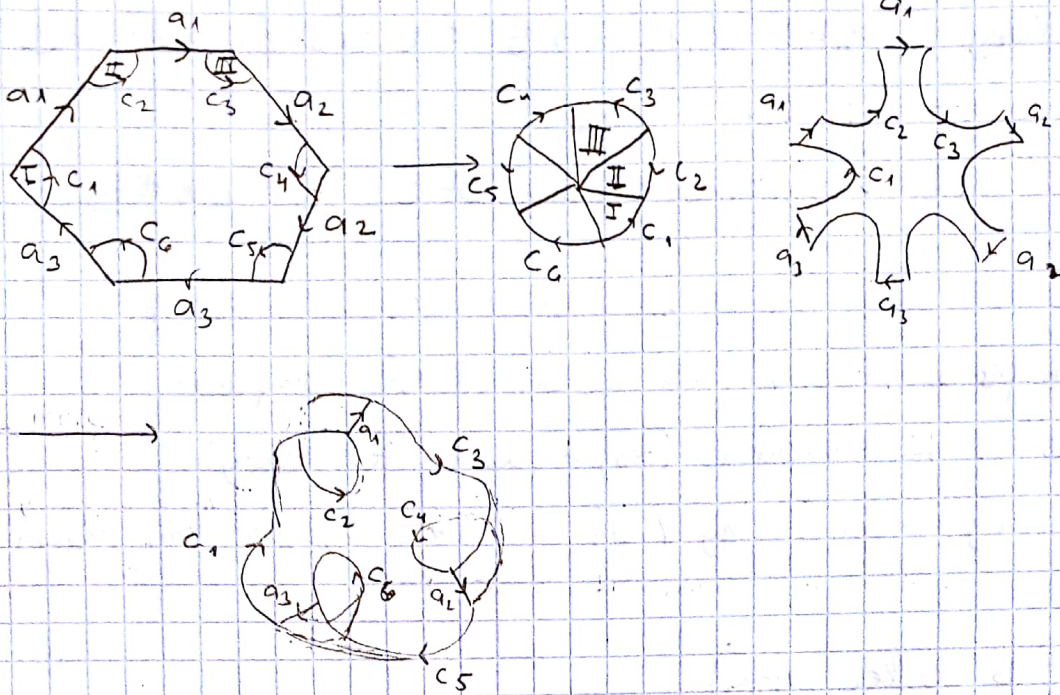




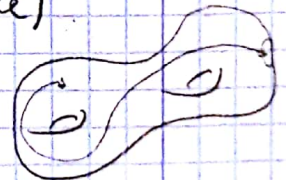
The arc complex:

S : Compact (connected) surface, possibly with boundary

V : non empty finite set of vertices $V = \{v_1, \dots, v_n\} \subseteq S$

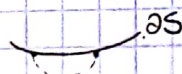
(Assume: at least one vertex from V in every boundary component arc:

An embedding of $[0,1]$ in S , with end points in V , but interior disjoint from V & ∂S (endpoints may coincide)
boundary



Essential arc:

not allowed:



does not cut S into two connected components one of which is a disc meeting V only at endpoints of the arc.

(*) An arc-system is a collection $\{\alpha_1, \dots, \alpha_k\}$ of essential arcs in S which are disjoint except perhaps the endpoints of the arc, & such that no two are "ambient isotopic" fixing V .



can't have 1 & 2 but can have 1 & 3

Def:

Ambient isotopy : $g, h : [0, 1] \rightarrow S$ are two embeddings.

$F : S \times [0, 1] \rightarrow S$ Continuous

$\forall t \in [0, 1] : F_t : \text{homeomorphism of } S$ $F_0 = \text{identity}$
 $F_t \circ g = h$

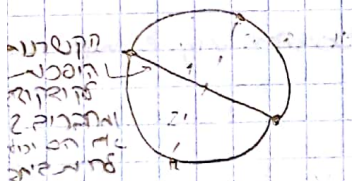
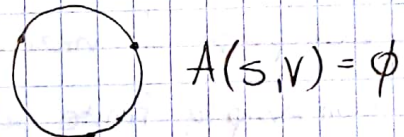
Def:

The arc-complex $A(S, V)$ is a simplicial complex (קומפלקס סימפליקלי) which has a k -simplex $[\alpha_0, \dots, \alpha_k]$ for every ambient-isotopy class fixing V of an arc system $\{\alpha_0, \dots, \alpha_{k+1}\}$ of $k+1$ arcs.

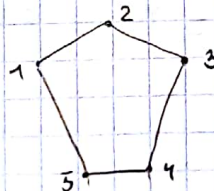
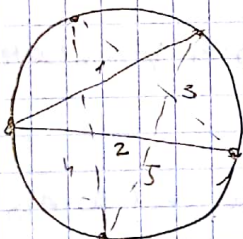
(faces : Sub collections)

Examples:

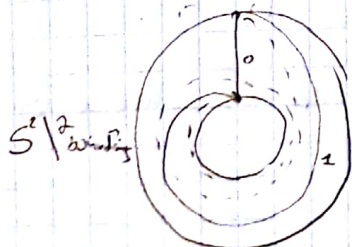
Disc, with 3 points on boundary;



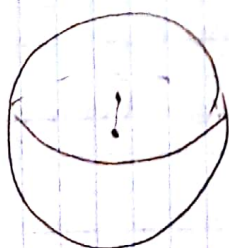
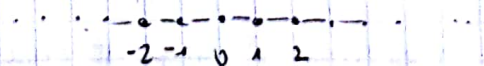
$A(S, V) =$



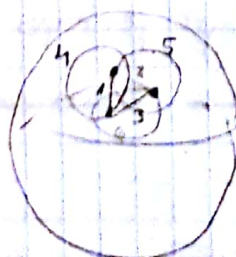
$A(S, V)$



$A(S, V) :$

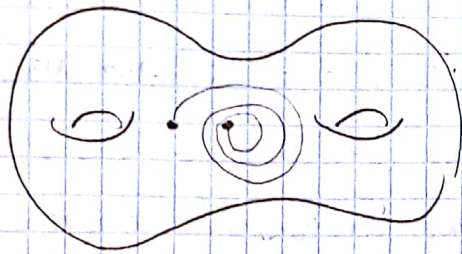


$A(S, V) :$



$A(S, V) :$





$$A(S, V) = ??$$

(*) The top-dimensional simplices in $A(S, V)$ correspond to triangulations of S . (two vertices / two edges of a triangle may coincide)

what is the dimension of a simplex?

(*) All maximal simplices of $A(S, V)$ have the same dimension
pf:

Let n be the number of triangles in a triangulation of S closed surface.

$$\chi(S) = \underbrace{|V|}_{\text{vertices}} - \underbrace{\frac{3n}{2}}_{\text{edges}} + \underbrace{n}_{\text{faces}} = |V| - \frac{n}{2}$$

← why does this prove the statement?

$$\Rightarrow n = 2(|V| - \chi(S))$$

for S with boundary - a similar computation shows that the number of arcs in a triangulation is fixed.

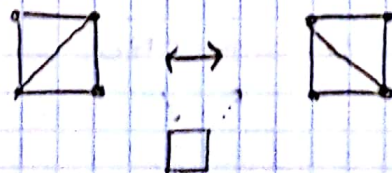
Thm (Moer, 85):

- $A(S, V)$ is either:
- 1) empty
 - or
 - 2) homeomorphic to a sphere, if S is a disc & $V \subseteq \partial S$
 - or
 - 3) Contractible (otherwise)

Hatcher, 1991: Much simpler proof. "Triangulation of Surfaces".

Corollary:

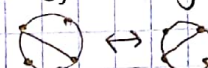
Every two triangulations of (S, V) are connected by elementary moves:



ff of Corollary:

top dimensional simplices of $A(S, V) = \text{triangulations}$.
 $\text{codim}-1$ simplices = triangulations \ a single arc. Here there
 are two options: 1) The arc belongs to two different triangles
 2) Belongs to one triangle.

need to show:

Any two top-dimensional simplices in $A(S, V)$ can be joined
 by a path passing only through open $\text{codim}-1$ / top-dim faces.
 (Unless A consists of disjoint points).
 this happens when: 

By Thm.: there is a path, but may pass through small-dimensional
 faces.

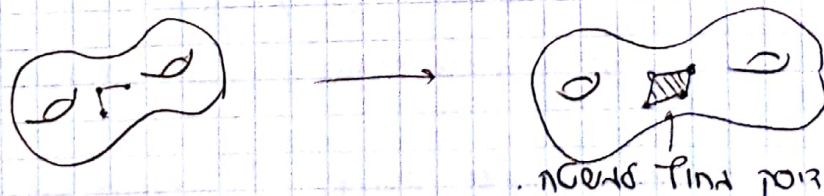
By induction on $\dim A$:

- if $\dim A = 1$ $\checkmark$ by thm.

- if $\dim A \geq 2$, say the path passes through some face F
 $\text{codim} \geq 2$. We show that the link of F is connected.
 because then we can construct a path going through
 faces of higher codim by induction.

The link of $F = \{ E \text{ face of } A \mid E \cap F = \emptyset, E \cup F \in A \}$

The link of F is identified with $A(S', V')$ where (S', V') is
 obtained from (S, V) by cutting S along the arcs of F .



$$\dim A(S', V') < \dim A(S, V)$$

$$= \text{codimension of } F - 1 \geq 1$$

$\Rightarrow \dim A(S', V')$ is connected.