Lecture 3: 21/3/2018

"Metric Theory"

Last time: Worked hard to show: $\alpha \notin \mathbb{Q}$ then $\alpha n^2 \bmod 1$ are <u>uniformly distributed</u>. (Weyl 1916)

<u>Open</u>: $\alpha 2^n$ unif dist mod 1?? $\Leftrightarrow$ α is <u>normal</u> base 2

if say $\alpha = \frac{1+\sqrt{5}}{2}, \pi, \ldots$ $\alpha = 0.a_1 a_2 a_3 \ldots$ $a_i \in \{0,1\}$ then any string eg 1011 appears with frequency $\frac{1}{2^4}$

<u>Thm</u>:

Let $\{a(n)\}$ be sequence of <u>distinct</u> integers $a(n) \in \mathbb{N}$, $a(n) \neq a(m)$, $n \neq m$

Then $\alpha\, a(n)$ is unif. dist. mod 1 for "almost all" α.

"almost all": The complement $\mathcal{C}$ has measure zero.

i.e $\forall \varepsilon > 0$ $\exists$ intervals $I_1, \ldots, I_N$ st (1) $\mathcal{C} \subseteq I_1 \cup \ldots \cup I_N$

(2) $\sum_i \ell(I_i) < \varepsilon$

<u>Example</u>: (1) for a.e α, αn^2 is U.D mod 1

(2) for a.e α, $\alpha \cdot 2^n$ U.D mod 1 $\leftrightarrow$ α is normed to base 2.
$2^n \neq 2^m$, $n \neq m$

(4) Almost all α are normed to base b for <u>all</u> $b \geq 2$

<u>Proof</u>:

Recall <u>Weyl's criterion</u>: $\{x_n\}$ U.D mod 1 $\quad e(z) := e^{2\pi i z}$

$\Leftrightarrow$ $S_k(N) := \frac{1}{N} \sum_{n \leq N} e(k x_n) \xrightarrow[N \to \infty]{} 0$, $\forall\, 0 \neq k \in \mathbb{Z}$

So want $\forall k \neq 0$, for a.e α. $S_k(\alpha, N) := \frac{1}{N} \sum_{n=1}^{N} e(k\alpha\, a(n)) \xrightarrow[N \to \infty]{} 0$

Suffices to fix K.

Suffices to show: For a.e α, $\sum_{N=1}^{\infty} |S(\alpha, N)|^2 < \infty$

Suffices to show: $\int_0^1 \sum_{N=1}^{\infty} |S(\alpha, N)|^2 \, d\alpha < \infty$

Fatou's Lemma $\Rightarrow$ $\int_0^1 \sum_{N=1}^{\infty} |S(\alpha, N)|^2 \, d\alpha \leq \sum_{N=1}^{\infty} \int_0^1 |S(\alpha, N)|^2 \, d\alpha$

$\left(\int^1 \liminf f_n(\alpha)\, d\alpha \leq \liminf \int^1 f_n(\alpha)\, d\alpha \right)$

$\Longrightarrow$ Suffices to show: $\sum_N \int_0^1 |S(\alpha,N)|^2 d\alpha < \infty$

Computation:

$$\int_0^1 |S(\alpha,N)|^2 d\alpha = \int_0^1 \left|\frac{1}{N}\sum_{n=1}^{N} e(\alpha\, a(n))\right|^2 d\alpha =$$

$$= \frac{1}{N^2}\int_0^1 \sum_{n=1}^{N} e(\alpha\, a(n)) \sum_{m=1}^{N} e(-\alpha\, a(m))\, d\alpha =$$

$$\overset{!}{=} \frac{1}{N^2}\sum_{n,m=1}^{N} \int_0^1 e^{2\pi i\, \alpha (a(n)-a(m))}\, d\alpha = \begin{cases} 1 & a(n)=a(m) \\ \left.\dfrac{e^{2\pi i \alpha (a(n)-a(m))}}{2\pi i (a(n)-a(m))}\right|_{\alpha=0}^{1} = 1-1=0 \end{cases}$$

$$\overset{!}{=} \frac{1}{N^2}\underset{a(n)=a(m) \Leftrightarrow n=m \text{ by assumption}}{\sum_{n,m=1}^{N}} 1 = \frac{1}{N^2}\sum_{n=1}^{N} 1 = \frac{1}{N}$$

So we get $\sum \int_0^1 |S(\alpha,N)|^2 d\alpha = \sum_{N=1}^{\infty} \frac{1}{N} = \infty$

We failed.

How to fix the problem:

1) pass to the subsequence of squares. $N_k = k^2$

Then: $\sum_k \int_0^1 |S(\alpha, N_k)|^2 d\alpha = \sum_k \frac{1}{N_k} < \infty$

$\Longrightarrow$ For almost all α, $S(\alpha, N_k) \longrightarrow 0$

2) For any $N>1$, $\exists!\, k$, s.t. $k^2 \le N < (k+1)^2$. Then show

$$|S(\alpha,N) - S(\alpha,k^2)| \xrightarrow[\text{for all } \alpha]{k\to\infty} 0 \Longrightarrow S(\alpha,N) \xrightarrow[N\to\infty]{} 0$$

$$|S(\alpha,N) - S(\alpha,k^2)| = \left|\frac{1}{N}\sum_{n=1}^{N} e(\alpha\, a(n)) - \frac{1}{k^2}\sum_{n=1}^{k^2} e(\alpha\, a(n))\right| =$$

$$= \left|\frac{1}{N}\sum_{n=k^2+1}^{N} e(\alpha\cdot a(n)) + \left(\frac{1}{N}-\frac{1}{k^2}\right)\sum_{n=1}^{k^2} e(\alpha\cdot a(n))\right| \le \quad \leftarrow |e(z)|=1,\ z\in\mathbb{R}$$

$$\le \frac{1}{N}\sum_{n=k^2+1}^{N} 1 + \left|\left(\frac{1}{N}-\frac{1}{k^2}\right)\right|\sum_{n=1}^{k^2} 1 \le \frac{1}{N}(N-k^2) + \left|\frac{N-k^2}{N\cdot k^2}\right| k^2$$

$$\le \frac{1}{N}(N-k^2) + \frac{N-k^2}{N} \le \quad \leftarrow k^2 \le N < (k+1)^2 = k^2+2k+1$$

$$0 \le N-k^2 \le 2k \le 2\sqrt{N}$$

$$\le 2\cdot \frac{2\sqrt{N}}{N} \ll \frac{1}{\sqrt{N}} \xrightarrow[\substack{N\to\infty \\ \text{as } k\to\infty}]{} 0$$

Quantitative Uniform distribution:

$\{x_n\} \subset [0,1)$ is unif. dist $\Longrightarrow$ $\{x_n\}$ is dense

$\longleftrightarrow$ every fixed interval $I \subset [0,1)$, $\exists n$ s.t $x_n \in I$

Question: shrinking intervals ??

$x_1, \dots, x_N$ want that every interval I of length $\frac{1}{m(N)}$ contains some p.t. $\{x_n\}$, $n \le N$, $m(N) \longrightarrow \infty$ $\left(\text{length}(I) \xrightarrow[N\to\infty]{} 0\right)$.

e.g. is there $n \le N$ s.t $\{\alpha n^2\} \in [0, \frac{1}{N^{1/2}}]$ note: need $\text{length}(I) \cdot \frac{1}{N}$

Discrepancy:

$X = \{x_n\} \subset [0,1)$. $D(N) := \sup\limits_{I \subset [0,1)} \left| \frac{1}{N} \#\{n \le N \mid x_n \in I\} - \text{length}(I) \right|$

Note: $D(N) \xrightarrow[N\to\infty]{} 0 \underset{\underset{(\text{check})}{\Longleftarrow}}{\Longrightarrow}$ uniform distribution

Note: If say we show $D(N) < \frac{1}{\sqrt{N}}$ then every interval of length $\gg \frac{1}{\sqrt{N}}$ will contain a p.t x_n, $n \le N$.

proof: Suppose $I_0 \cap \{x_n \mid n \le N\} = \phi$

$$\frac{1}{\sqrt{N}} > D(N) \ge \left| \frac{1}{N} \underbrace{\#\{n \le N \mid x_n \in I_0\}}_{=0} - \text{length}(I_0) \right| = \text{length}(I_0)$$

$\Longrightarrow \text{length}(I_0) < \frac{1}{\sqrt{N}}$. So if $\text{length}(I_0) \ge \frac{1}{\sqrt{N}}$ then $\exists n \le N$ s.t $x_n \in I_0$.

Exercise: $\frac{1}{N} \le D(N) \le 1$

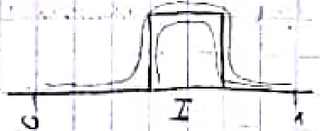
Erdös - Turan inequality:

$\forall K \ge 1$, $D(N) \le \frac{1}{K+1} + 3 \sum_{k=1}^{K} \frac{1}{k} |S(k,N)|$

$S(k,N) = \frac{1}{N} \sum_{n=1}^{N} e(k x_n)$: normalized weyl sum.

$\left(\text{recall: UD} \iff S(k,N) \xrightarrow[\forall k \ne 0.]{N\to\infty} 0 \right)$

Crude attempt to bound discrepancy:



$$\frac{1}{N}\#\{n\le N \mid x_n\in I\} = \frac{1}{N}\sum_{n=1}^{N}\mathbb{1}_I(x_n)$$

Now try to replace $\mathbb{1}_I$ by its Fourier Series.,

$$\mathbb{1}_I(x) \overset{\text{"="}}{\underset{L^2}{}} \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x}, \quad a_n = \langle \mathbb{1}_I, e_n\rangle = \int_0^1 \mathbb{1}_I(x) e^{-2\pi i n x}\,dx =$$

$n\neq 0$, $I=[\alpha,\beta]$

$$= \int_\alpha^\beta 1\cdot e^{-2\pi i n x}\,dx = \frac{e^{-2\pi i n\beta} - e^{-2\pi i n \alpha}}{-2\pi i n}$$

$$n\neq 0: \quad a_n = \frac{e^{-2\pi i n\alpha} - e^{-2\pi i n\beta}}{2\pi i n}, \quad a_0 = \text{length}(I)$$

Pretending that everything converges nicely

$$\frac{1}{N}\sum_{n=1}^{N}\mathbb{1}_I(x_n) = \frac{1}{N}\sum_{n=1}^{N}\sum_{k=-\infty}^{\infty} a_k e_k(x_n) =$$

$$\text{"="} \sum_{k=-\infty}^{\infty} a_k \cdot \underbrace{\frac{1}{N}\sum_{n=1}^{N} e_k(x_n)}_{S_k(N)} = \underset{\overset{\shortparallel}{\text{length}(I)}}{a_0} + \sum_{k\neq 0} \frac{*}{k} S_k(N)$$

Ideally take $N\to\infty$ & change $\lim_N \sum_k = \sum_k \lim_N$

Goal: Study discrepancy of various sequences.

Main example:

$x_n = \alpha n \bmod 1$, $\alpha\notin\mathbb{Q}$ U.D

Lets try to use Erdös-Turan. Need

$$S_k(N) = \frac{1}{N}\sum_{n=1}^{N} e(k\alpha n) = \frac{1}{N}\frac{e(k\alpha) - e((N+1)k\alpha)}{1-e(k\alpha)}, \quad e(k\alpha) = e^{2\pi i k\alpha}.$$

$$S_k(N) \le \frac{1}{N}\frac{2}{|\sin(\pi k\alpha)|} \ll \frac{1}{N}\frac{1}{\|k\alpha\|}, \quad \|z\| := \text{dist}(z,\mathbb{Z})$$

E-T:

$$D(N) \ll \frac{1}{K} + \frac{1}{N}\sum_{k=1}^{K}\frac{1}{k\|k\alpha\|} \quad (\alpha\notin\mathbb{Q})$$

(not clear how to bound this)

Bad example: Suppose α has ∞ many approximations

$$\left|\alpha - \frac{a_i}{k_i}\right| < \frac{1}{k_i 2^{k_i}}$$

then $|k_i\alpha - a_i| < \frac{1}{2^{k_i}} \Rightarrow \|k_i\alpha\| < \frac{1}{2^{k_i}} \Rightarrow \sum_{k=1}^{k_i}\frac{1}{k}\frac{1}{\|k\alpha\|} > \frac{1}{k_i\|k_i\alpha\|} > \frac{2^{k_i}}{k_i}$

$\Longrightarrow$ Will not do better than $D(N) < \frac{1}{\log N}$

Example: (Liouville) $\alpha = 0.110001000\ldots 010\ldots$ (digits 1 at positions $2!$, $3!$, $4!$)

$$\left|\alpha - \frac{a_n}{10^{n!}}\right| < \frac{1}{(10^{n!})^{n+1}} \qquad \left|\alpha - \frac{a_n}{q_n}\right| < \frac{1}{q_n^{n+1}}, \quad q_n = 10^{n!}$$

$\frac{1}{10^{n\cdot n!}}$

Def:

α is badly approximable (or of "constant type") if $\exists c > 0$.

$$\left|\alpha - \frac{a}{q}\right| \geq \frac{c}{q^2} \quad \forall a, q \geq 1, \ \frac{a}{q} \neq \alpha$$

Ex: $\alpha \in \mathbb{Q}$ then $\exists c$. s.t $\left|\alpha - \frac{a}{q}\right| > \frac{c}{q}$, $\frac{a}{q} \neq \alpha$.

Dirichlet: $\alpha \notin \mathbb{Q}$ then $\exists \infty$ many co-prime a, q, s.t.

$$\left|\alpha - \frac{a}{q}\right| < \frac{1}{q^2}$$

$$\left(Q \geq 1, \ \exists q \leq Q \ \text{ s.t } \left|\alpha - \frac{a}{q}\right| < \frac{1}{qQ}\right)$$

Prop.: $1 \leq D \neq \square$ integer.

$\alpha = \sqrt{D}$ is a quadratic irrationality, then α is badly approximable

$$\left|\alpha - \frac{a}{q}\right| \geq \frac{1}{3\sqrt{D}} \cdot \frac{1}{q^2}$$

Proof: ($\alpha = \sqrt{D}$)

WLOG, $\left|\alpha - \frac{a}{q}\right| < \frac{1}{10}$. Let $f(x) = x^2 - D$ & consider

$$\left|f\left(\frac{a}{q}\right) - f(\sqrt{D})\right| = \left|\left(\frac{a}{q}\right)^2 - D\right| = \left|\frac{a^2 - Dq^2}{q^2}\right| \geq \frac{1}{q^2}$$

$f(\sqrt{D}) = 0$, $\left(\frac{a}{q}\right)^2 - D = \left(\frac{a}{q} - \sqrt{D}\right)\left(\frac{a}{q} + \sqrt{D}\right)$

since $D \neq \left(\frac{a}{q}\right)^2 \to 0 \neq a^2 - Dq^2 \in \mathbb{Z}$

$$\left|\frac{a}{q} - \sqrt{D}\right| \left|\frac{a}{q} + \sqrt{D}\right| \geq \frac{1}{q^2}$$

$$\left|\frac{a}{q} - \sqrt{D}\right| \geq \frac{1}{q^2} \frac{1}{\frac{a}{q} + \sqrt{D}} > \frac{1}{3\sqrt{D}} \cdot \frac{1}{q^2}$$

$\blacksquare$

Liouville's Thm.

α is real algebraic of degree $d \geq 2$, then $\exists c(\alpha) > 0$ s.t

$$\left|\alpha - \frac{a}{q}\right| \geq \frac{c(\alpha)}{q^d}$$

$c(\alpha)$ is explicit (GOOD)

(BAD)

Open Problem: Show that an algebraic number of degree $d \geq 3$ is Not badly approx.

Facts:

- α is badly approx $|\alpha - \frac{a}{q}| > \frac{c}{q^2}$ $\Longleftrightarrow$ Continued fraction expansion of $\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_{2} + \dots}}$ has bounded a_i

NB: α = quadratic irrationality $\Longleftrightarrow$

Roth's Thm.: (dont know anything about the dependance of c on α, ε)

α algebraic, then $\forall \varepsilon > 0$ $\exists$ $c(\alpha, \varepsilon) > 0$ $|\alpha - \frac{a}{q}| > \frac{c}{q^{2+\varepsilon}}$

NB: $\{$Badly approximable $\alpha\}$ is of measure zero.

NB: "Diophantin α" $\left(\exists c, d \; |\alpha - \frac{p}{q}| > \frac{c}{q^d}\right)$ has full measure.

Goal: Discrepancy of $\{\alpha n^2\}$, α is badly approx. (eg $\alpha = \frac{1+\sqrt{5}}{2}$

Saw: $\forall \alpha$ $D(\{\alpha n\}; N) \ll \frac{1}{K} + \frac{1}{N} \sum_{k=1}^{K} \frac{1}{k \|k\alpha\|}$

Thm.:

α is badly approx then $D(\{\alpha n\}, N) \ll \frac{(\log N)^2}{N}$

"Low Discrepancy"

Prop.: α is badly approx.

Let $A(t) := \sum_{k \leq t} \frac{1}{\|k\alpha\|}$, $G(K) = \sum_{k=1}^{K} \frac{1}{k\|k\alpha\|}$ then:

$A(t) \ll t \log t$, $G(K) \ll (\log K)^2$

Corollary:

α is badly approx.

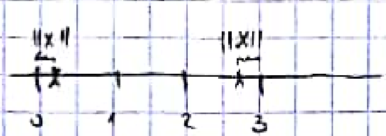
$D(N) \ll \frac{1}{K} + \frac{1}{N} (\log K)^2 \underset{K = N}{\ll} (\log N)^2 / N$

Cor.:

Every interval of length $> (\log N)^2/N$ will contain some fractional part $\{\alpha n\}$, $n \leq N$

Lemma:

α badly approx. then $\big|\, \|m\alpha\| - \|n\alpha\| \,\big| \gg \frac{1}{j}$ if $1 \le m \ne n \le j$



$$\|m\alpha\| = \begin{cases} M - m\alpha \\ m\alpha - M \end{cases}$$

$$\|m\alpha\| - \|n\alpha\| = \begin{cases} M - m\alpha \\ m\alpha - M \end{cases} - \begin{cases} N - n\alpha \\ n\alpha - N \end{cases} = \alpha(\pm m \pm n) \pm \text{integer}$$

$$\big|\, \|m\alpha\| - \|n\alpha\| \,\big| \ge \|\alpha(m \pm n)\| \gg \frac{1}{m+n} > \frac{1}{2j}$$

$$|q\alpha - a| > \frac{1}{q} \iff \left|\alpha - \frac{a}{q}\right| > \frac{1}{q^2} \quad \alpha \text{ badly approx}$$

Now bound $A(t) = \sum_{k \le t} \frac{1}{\|k\alpha\|}$.

divide the interval $[0, \frac{1}{2}]$ into intervals of length $\approx \frac{1}{t}$. each such interval will contain at most <u>one</u> $\|k\alpha\|$, $k \le t$.

$$\sum_{k \le t} \frac{1}{\|k\alpha\|} \asymp \sum_{j=1}^{t} \sum_{\substack{k \le t \\ \text{s.t. } \frac{j}{t} < \|k\alpha\| \le \frac{j+1}{t}}} \frac{1}{\|k\alpha\|} \le$$

note $j \ne 0$ because $\|k\alpha\| \asymp \{k\alpha\} > \frac{1}{k} \ge \frac{1}{t}$

$$\le \sum_{j=1}^{t} \frac{1}{j/t} \underbrace{\#\left\{k \le t \mid \|k\alpha\| \in \left[\frac{j}{t}, \frac{j+1}{t}\right]\right\}}_{\le 1 \text{ Lemma}} \ll t \sum_{j=1}^{t} \frac{1}{j} \ll t \log t$$

Exercise:

Show $A(t) \ll t \log t$ then $G(K) \ll (\log K)^2$. (Summation by parts).

Showed:

$$D(\{\alpha n\}; N) \ll \frac{(\log N)^2}{N} \quad \text{if } \alpha \text{ is badly approx.}$$

$\Longrightarrow$ every interval of length $> \frac{(\log N)^2}{N}$ will contain some $\{\alpha n\}$; $n \le N$

To think: discrepancy of $\{\alpha n^2\}$