Lecture 2

5/3/2017

Classification of surfaces:

Thm. [Dehn-Heegard 1907]:

The following is the complete list of connected, closed surfaces (up to homeo):

1) S^2
2) $\underbrace{T \# T \# \dots \# T}_{\text{times } n}$ $\forall n \geq 1$
3) $\underbrace{P \# \dots \# P}_{n \text{ times}}$ $\forall n \geq 1$ $\quad (P = \mathbb{R}P^2)$

Proof:

Part 1: every 2 in the list are not homeomorphic.

Argument 1: We saw:

$$\left.\begin{aligned} \chi(S^2) &= 2 \\ \chi(\underbrace{T \# \dots \# T}_{n \text{ times}}) &= 2 - 2n \end{aligned}\right\} \text{orientable}$$

$$\chi(\underbrace{P \# \dots \# P}_{n \text{ times}}) = 2 - n \quad \leftarrow \text{non-orientable}$$

Orientability:

A surface is orientable if it does not contain an embedded Möbius band.

$\iff$ $\exists$ continuous choice of a normal/orientation at every point.

Fact:

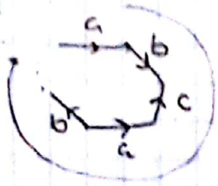
Euler constant & orientability are topological invariants.

$\Rightarrow$ Every 2 in the list are not homeomorphic.

Argument 2: using fundamental group.

Canonical representation of these surfaces

Assume that S_1 represented by:

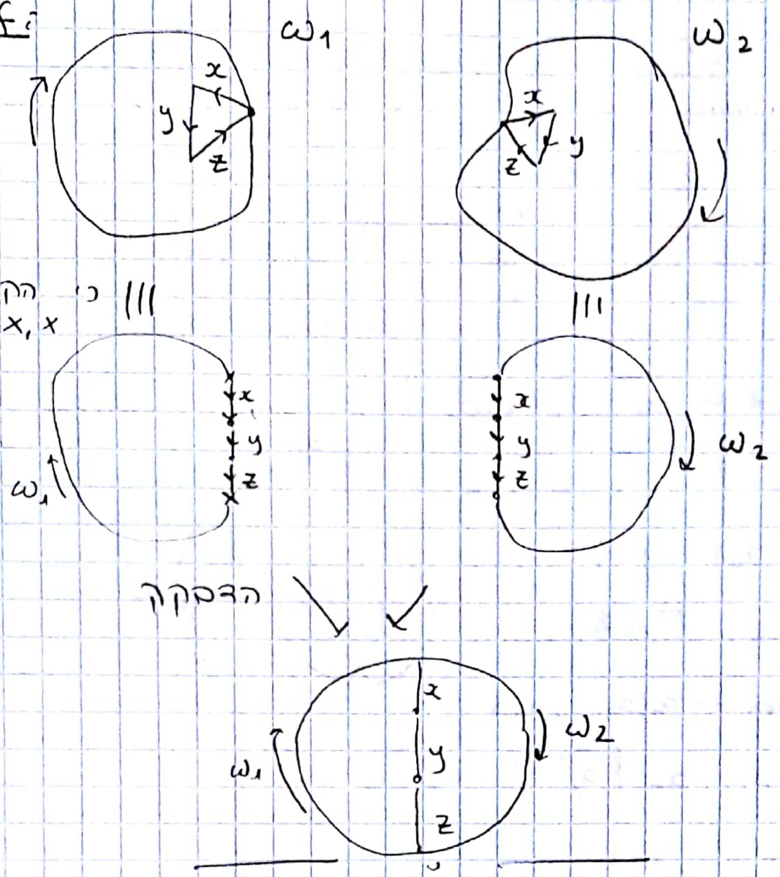


$w_1 = abc^{-1}a^{-1}b \dots$

Assume that S_2 represented by w_2

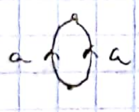
& w_1, w_2 are disjoint (אין אותיות זהות בשני המילים) then $S_1 \# S_2$ is represented by $\omega_1 \omega_2$

Proof:

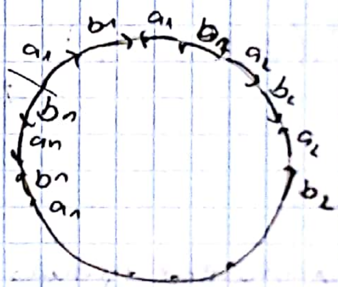


Canonical representations:

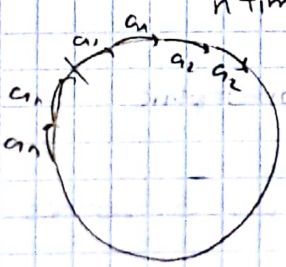
S^2 :



$\underbrace{T\#\dots\#T}_{n \text{ times}}$:



$\underbrace{P\#\dots\#P}_{n \text{ times}}$.



$T\#P = ?$

$\chi(T\#P) = \chi(T) + \chi(P) - 2 = -1$

orientable? no. $\Longrightarrow$ $T\#P = P\#P\#P$

Thm. (Seifert - Van Kampen):

Let (X, x_0) be a pointed, path-connected top space. Assume $X = A \cup B$ with A, B open, $x_0 \in A, B$, $A \cap B$ path-connected. Then:

$$\pi_1(X, x_0) = \frac{\pi_1(A, x_0) * \pi_1(B, x_0)}{\pi_1(A \cap B, x_0)} = \frac{\langle \pi_1(A, x_0), \pi_1(B, x_0) \rangle}{\langle\langle i_*^{A\cap B \to A}(\gamma) = i_*^{A\cap B \to B}(\gamma) \rangle\rangle \ \forall \gamma \in \pi_1(A \cap B}$$

$\pi_1(S^2)=\{0\}$

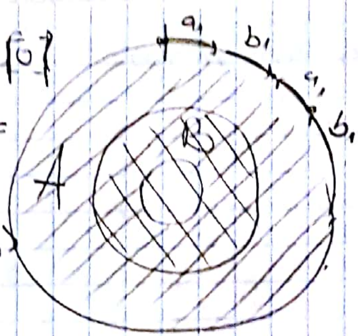
$\pi_1(\underbrace{T\#\ldots\#T}_{n \text{ times.}})=?$

$\pi_1(B,x_0)=\{0\}$

$\pi_1(A,x_0)=\pi_1(\infty\!\!\infty\,\cdot)=$
2n לולאות

$=F_{2n}=\langle a_1,b_1,\ldots,a_n,b_n\rangle$

החבורה החופשית על 2n יוצרים



$$\pi_1(A\cap B,x_0)\cong\mathbb{Z}=\langle\gamma\rangle$$

$i_*^{A\cap B\to B}(\gamma)=e \quad , \quad i_*^{A\cap B\to A}(\gamma)=\underbrace{a_1b_1a_1^{-1}b_1^{-1}}\cdots\underbrace{a_nb_na_n^{-1}b_n^{-1}}$

$=[a_1,b_1]\cdots[a_n,b_n]$

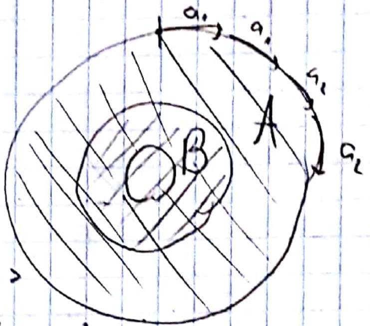
$\Longrightarrow \pi_1(T\#\ldots\#T)=\langle a_1,b_1,\ldots,a_n,b_n \mid [a_1,b_1]\cdots[a_n,b_n]=e\rangle$

$\pi_1(P\#\ldots\#P)=?$

$\pi_1(B,x_0)=\{0\}$

$\pi_1(A,x_0)=\pi_1(\infty\!\!\infty,\cdot)=$

$=F_n=\langle a_1,a_2,\ldots,a_n\rangle$



$\pi_1(P\#\ldots\#P)=\langle a_1,a_2,\ldots,a_n \mid a_1^2a_2^2\cdots a_n^2=e\rangle$

$\mathrm{Abel}(\pi_1(T\#\ldots\#T))=\mathbb{Z}^{2n}$

$\mathrm{Abel}(\pi_1(P\#\ldots\#P))=\underset{a_1\mathbb{Z}+a_2\mathbb{Z}+\ldots+a_n\mathbb{Z}}{\mathbb{Z}^n}/\langle 2a_1+2a_2+\ldots+2a_n=0\rangle=\mathbb{Z}^{n-1}\times\mathbb{Z}_2$

$a_1,a_2,\ldots,a_{n-1},a_1+a_2+\ldots+a_n$

<u>Corollary</u>:

The fundamental groups are not isomorphic.

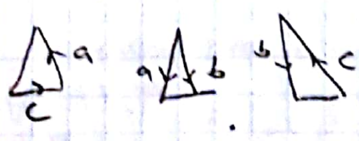
<u>Part II of proof</u>: Why every surface is homeo. to one of the surfaces in the list.

<u>Thm. (Triangulation, Radó 1925)</u>:

Every surface can be triangulated.

(a 2-dim simplical complex)

I.e. every surface can be obtained as a finite union of disjoint triangles with edges identified in pairs.



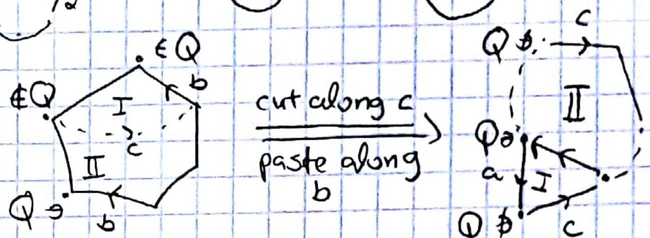
$\Rightarrow$ Every Surface can be obtained as a polygon with $2n$ edges, that are glued in pairs.

Step 1: Reduction to a single Vertex (or to $a\bigcirc a$)

If there is more then one equivalence class of vertices: let Q one of these classes.

- If $|Q| = 1$

- If $|Q| \geq 2$



this reduces $|Q|$ by 1