

Reminder:

Let $X = \{x_n\}$ be a sequence of reals. We say X is uniformly distributed mod 1 if $\forall$ every subinterval $I \subset [0, 1)$

$$\frac{1}{N} \# \{ \underbrace{\{x_n\}}_{\text{fractional part}} \in I \mid n \leq N \} \xrightarrow{N \rightarrow \infty} \text{length}(I)$$

$\Rightarrow$ The fractional parts $\{x_n\}$ are dense in $[0, 1)$.

The Converse is false. $x_n = \log n$ then we saw that they are not u.d. exercise $\{\log n\}$ are dense in $[0, 1)$.

Weyl's Criterion:

$$\forall 0 \neq k \in \mathbb{Z} \quad \frac{1}{N} \sum_{n=1}^N e(kx_n) \longrightarrow 0 \quad e(z) := e^{2\pi iz}$$

$\Leftrightarrow$ Fractional part of x_n are u.d.

Example:

$$x_n = \alpha n, \quad \alpha \in \mathbb{Q} \Rightarrow \text{u.d. mod 1}$$

Proof:

$$\text{Show } \sum_{n=1}^N e(k\alpha n) = o(N) \quad \forall k \neq 0.$$

$$\text{Indeed: } \sum_{n=1}^N e(k\alpha n) = \sum_{n=1}^N e(k\alpha)^n = \frac{e(k\alpha) - e(k\alpha(N+1))}{1 - e(k\alpha)}$$

$$x + x^2 + \dots + x^N = \frac{x - x^{N+1}}{1 - x} \quad \& \alpha \in \mathbb{Q} \text{ so } e^{2\pi i k \alpha} \neq 1 \quad \forall k \neq 0.$$

$$\left| \sum_{n=1}^N e(k\alpha n) \right| \leq \frac{|e(k\alpha)| + |e(k\alpha(N+1))|}{|1 - e(k\alpha)|} \quad (e^{it} \neq 1 \quad \forall t \in \mathbb{R})$$

$$1 - e(k\alpha) = 1 - e^{2\pi i k \alpha} = e^{\pi i k \alpha} (e^{-\pi i k \alpha} - e^{\pi i k \alpha}) = -2i \sin(\pi k \alpha)$$

$$\text{So } \left| \frac{1}{N} \sum_{n=1}^N e(k\alpha n) \right| < \frac{1}{N} \text{const}(k\alpha) \xrightarrow{N \rightarrow \infty} 0$$

How to prove Weigl's Criterion:

Observe $U.D. \iff \underbrace{\frac{1}{N} \sum_{n=1}^N \mathbb{1}_I(x_n)}_{\# \{n \leq N \mid x_n \in I\}} \longrightarrow \underbrace{\int_0^1 \mathbb{1}_I(x) dx}_{= \text{length}(I)}$

$$\mathbb{1}_I(x) = \begin{cases} 1 & x \in I \\ 0 & \text{else} \end{cases} \quad \forall I \subset [0,1] \text{ interval.}$$

Lemma: $(*) \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(x_n) = \int_0^1 f(x) dx$

The following are equivalent:

- obvious. $\left(\begin{array}{l} 1) \{x_n\} \text{ is U.D. } (*) \text{ holds } \forall f = \mathbb{1}_I \\ 2) (*) \text{ holds for all continuous } f \text{ on } [0,1] \\ 3) (*) \text{ holds for all Riemann-integrable } f \end{array} \right.$

1) $\implies$ 3):

$\forall \epsilon > 0$

Recall that f is Riemann-integrable iff $\exists$ step function -

(finite linear comb. of indicator function of intervals) S_+, S_- s.t.

1) $S_- \leq f \leq S_+$

2) $\int (S_+ - S_-) < \epsilon$

Now $U.D. \iff (*)$ for indicator functions $\implies (*)$ for step function

$$(S = \mathbb{1}_I + 2\mathbb{1}_J \quad \frac{1}{N} \sum_{n=1}^N S(x_n) = \frac{1}{N} \sum_{n=1}^N \underbrace{\mathbb{1}_I(x_n)}_{\int \mathbb{1}_I} + 2 \frac{1}{N} \sum_{n=1}^N \underbrace{\mathbb{1}_J(x_n)}_{\int \mathbb{1}_J} \longrightarrow$$

$$\longrightarrow \int (\mathbb{1}_I + 2\mathbb{1}_J) = \int S(x) dx.$$

$$\begin{aligned} \frac{1}{N} \sum_{n=1}^N f(x_n) - \int f(x) &= \frac{1}{N} \sum_{n=1}^N \underbrace{(f(x_n) - S_+(x_n))}_{\leq 0} + \frac{1}{N} \sum_{n=1}^N S_+(x_n) \\ &= \underbrace{\int_0^1 S_+(x) dx}_I + \underbrace{\int_0^1 (S_+(x) - f(x)) dx}_{II} \leq \end{aligned}$$

$$\leq I + II$$

Show: $\exists N(\epsilon)$ s.t. $\forall N \geq N(\epsilon), |I| < \epsilon$ - ok, because
by (1), (*) holds for step functions like S_+ .

$$II = \int_0^1 (S_+ - f) < \int_0^1 (S_+ - S_-) < \epsilon$$

Hence: $\frac{1}{N} \sum_{n=1}^N f(x_n) - \int_0^1 f(x) < 2\epsilon$

Likewise replace S_+ by S_- will get: $\frac{1}{N} \sum f(x_n) - \int f > -2\epsilon$

$\Rightarrow \forall \epsilon > 0, \exists N(\epsilon)$ s.t. $\forall N \geq N(\epsilon)$

$$\left| \frac{1}{N} \sum_{n=1}^N f(x_n) - \int_0^1 f(x) dx \right| < 2\epsilon \quad \text{i.e. (*) holds for } f.$$

2) $\Rightarrow$ 1):

Approximate indicator functions by continuous as before?



Weyl's Criterion:

(*) holds for all $k \in \mathbb{Z}$

$$\frac{1}{N} \sum_{n=1}^N e^{ikx_n} \longrightarrow \int_0^1 e^{ikx} dx = \begin{cases} 1 & k=0 \\ 0 & k \neq 0 \end{cases}$$

By linearity, Weyl's Criterion $\Leftrightarrow$ (*) for all trigonometric pols.
 $T(x) = \sum_{k=-K}^K c_k e^{2\pi i k x}$

claim:

(*) holds $\forall$ trig. polys $\Leftrightarrow$ (*) holds $\forall$ continuous fn. $\overset{\text{lemma}}{\Leftrightarrow}$ UD

Weierstrass' Approximation thm.:

Every continuous fn. on $[0, 1]$ is uniformly approx by trig polys.

$\forall \epsilon > 0, \exists T_\epsilon$ trig. pol s.t. $\|f - T_\epsilon\|_\infty < \epsilon$

$$\sup_{x \in [0, 1]} |f(x) - T_\epsilon(x)|$$

So take f cont., $\epsilon > 0, T_\epsilon$ as above

$$\begin{aligned} \frac{1}{N} \sum_{n=1}^N f(x_n) - \int_0^1 f(x) dx &= \underbrace{\frac{1}{N} \sum_{n=1}^N (f(x_n) - T_\epsilon(x_n))}_{\text{I}} + \\ &+ \underbrace{\frac{1}{N} \sum_{n=1}^N T_\epsilon(x_n) - \int_0^1 T_\epsilon(x)}_{\text{II}} + \underbrace{\int_0^1 (T_\epsilon(x) - f(x)) dx}_{\text{III}} \end{aligned}$$

Take $N(\epsilon)$ s.t. $\forall N \geq N(\epsilon)$, $|I| < \epsilon$ (OK because assuming (*) holds for trig poly c)

$$|II| \leq \int |T_\epsilon(x) - f(x)| dx < \epsilon$$

$$|III| \leq \frac{1}{N} \sum_{n=1}^N \underbrace{|f(x_n) - T_\epsilon(x_n)|}_{< \epsilon} < \epsilon$$

$$\Rightarrow \left| \frac{1}{N} \sum f(x_n) - \int f \right| < 3\epsilon \quad \forall N \geq N(\epsilon)$$

Thm. (Weyl, 1916)

The sequence αn^2 is U.D. mod 1 if $\alpha \notin \mathbb{Q}$

Start of Proof:

Need to show:

$$\forall k \geq 0 \quad S(N) = \sum_{n=1}^N e(\alpha k n^2) = o(N)$$

Weyl's differencing trick:

$$\begin{aligned} |S(N)|^2 &= \sum_{n=1}^N e(\alpha k n^2) \sum_{m=1}^N e(\alpha k m^2) = \overline{e(z)} = e(-z) \text{ if } z \in \mathbb{R} \\ &= \sum_{n=1}^N e(\alpha k n^2) \sum_{m=1}^N e(-\alpha k m^2) = \sum_{n=1}^N \sum_{m=1}^N e(\alpha k (n^2 - m^2)) = \\ &= \sum_{m=n=1}^N 1 + \sum_{m \neq n} e(\alpha k (n^2 - m^2)) = 1 + e(z) + e(-z) = 2\operatorname{Re}(e(z)) \\ &= N + 2\operatorname{Re} \sum_{1 \leq m \neq n \leq N} e(\alpha k (n^2 - m^2)) \end{aligned}$$

Write $n = m + h$, $1 \leq h \leq N - m$, $1 \leq m \leq N - 1$

or $1 \leq h \leq N - 1$, $1 \leq m \leq N - h$

$$n^2 - m^2 = (n - m)(n + m) = h(2m + h) = h^2 + 2hm$$

$$\text{off} = \sum_{1 \leq m < n \leq N} e(\alpha k (n^2 - m^2)) = \sum_{h=1}^{N-1} e(\alpha k h^2) \sum_{m=1}^{N-h} e(\alpha k 2h \cdot m)$$

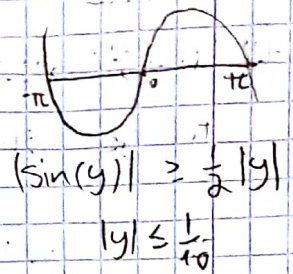
a geometric progression

$$|\text{off}| \leq \sum_{h=1}^{N-1} \left| \sum_{m=1}^{N-h} e(\alpha k 2h \cdot m) \right|$$

↑
irrational $\Rightarrow$ this is never 1.

want $o(N^2)$

$$\left| \sum_{m=1}^{N-h} e\left(\frac{2\pi k \cdot h \cdot m}{\delta}\right) \right| \leq \frac{1}{|\sin(\pi \delta h)|}$$



$$|off| \leq \sum_{k=1}^{N-1} \min\left(\frac{1}{|\sin(\pi \delta h)|}, N\right)$$

Proposition:

$\delta \notin \mathbb{Q}$ then $\sum_{h=1}^{N-1} \min\left(N, \frac{1}{\|\delta h\|}\right) = o(N^2)$ where $\|x\| := \text{dist}(x, \mathbb{Z})$

(This will prove Weyl's Criterion)

Idea: replace δ by a rational approximation.

Dirichlet's Lemma:

Given $x \in \mathbb{R}$, $N \geq 1 \exists a \in \mathbb{Z} \ 1 \leq q \leq N$ s.t. $|x - \frac{a}{q}| < \frac{1}{qN}$.

Moreover, if $x \notin \mathbb{Q}$ then $q \rightarrow \infty$ as $N \rightarrow \infty$.

Proof:

Divide $[0, 1)$ into N subintervals of length $\frac{1}{N}$

consider the $N+1$ pts $0, \{qx\}, \{2qx\}, \dots, \{Nqx\} \in [0, 1)$

$\Rightarrow \exists$ subinterval which contains $\{mx\}, \{nx\}$, $0 \leq m \neq n \leq N$

$\{y\} := y - \lfloor y \rfloor$ integer

$$0 \leq \{mx\} - \{nx\} < 1/N$$

$$\frac{mx - \lfloor mx \rfloor}{m_1} - \frac{nx - \lfloor nx \rfloor}{m_2} \quad m_1, m_2 \in \mathbb{Z}$$

$$0 \leq \frac{(m-n)x}{q} - (M_1 - M_2) < \frac{1}{N}$$

Say $q := m-n > 0$ then $1 \leq q \leq N$, set $a := M_1 - M_2 \in \mathbb{Z}$

$$0 \leq qx - a < \frac{1}{N} \rightarrow 0 < x - \frac{a}{q} < \frac{1}{qN}$$

Moreover, If $x \notin \mathbb{Q}$ then $q \rightarrow \infty$ with $N \rightarrow \infty$

else, $\exists$ infinitely N 's s.t. $(\text{say}) 0 < |x - \frac{a}{q}| < \frac{1}{qN} \xrightarrow{N \rightarrow \infty} 0$

Remarks:

1) Proof is not constructive. Can use continued fractions to get such approximations.

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}} \quad a_0 \in \mathbb{Z}, a_1, a_2, a_3, \dots \geq 1$$

take n 'th convergent $\frac{p_n}{q_n} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots + \frac{1}{a_n}}}$ then:

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2}$$

Examples:

$$\rightarrow x = \frac{1+\sqrt{5}}{2} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}} \quad \frac{p_n}{q_n} = \frac{F_{n+1}}{F_n}, \quad F_n = n\text{'th Fibonacci } 0, 1, 1, 2, 3, 5, 8, 13, 21, \dots$$

$\rightarrow x = \text{quadratic irrationality}$

$$\frac{a + \sqrt{D}}{b} \iff \text{periodic cont. frac. expansion}$$

$$D, a, b \in \mathbb{Z}$$

Quadratic irrationalities are BADLY approximable by rationals:

$$\exists c = c(x). \quad \left| x - \frac{p}{q} \right| \geq \frac{c}{q^2} \quad \forall p, q.$$

Examples

$$\rightarrow x = 0.\overset{1!}{1}\overset{2!}{1}0000\overset{6=3!}{1}0 \dots \overset{24=4!}{0}\overset{1}{1}0 \dots \overset{120=5!}{0}\overset{1}{1}0 \dots$$

(Liouville) is extremely well approximable:

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n^{n+1}}, \quad q_n = 10^n$$

Liouville:

If x is real algebraic of degree $d \geq 1$ then $\exists c(x) > 0$ s.t.
if $x \neq \frac{p}{q}$ then $\left| x - \frac{p}{q} \right| \geq \frac{c}{q^d} \implies x$ is transcendental

Thue, Siegel, Dyson, ..., Roth (1950):

x is algebraic irrational, $\forall \epsilon > 0, \exists c(x, \epsilon) > 0$ s.t. $\left|x - \frac{p}{q}\right| \geq \frac{c}{q^{2+\epsilon}}$

Application:

Show finiteness of # of solutions of Diophantine eqs.

Bad example: Pell's equation $x^2 - 2y^2 = 1$ has ∞ many integer sols. Start from $(3, 2)$ $(3 + 2\sqrt{2})^n = x_n + \sqrt{2}y_n$ then $x_n^2 - 2y_n^2 = 1$

Thue's equation:

$$x^3 - 2y^3 = 1$$

And Thue 1909 only finitely many integer sols.

Roth's Thm: $\Rightarrow \left|\sqrt[3]{2} - \frac{p}{q}\right| > \frac{c}{q^{5/2}}$ ($\frac{5}{2} < 3$)

$$\frac{1}{|y|^3} = \left|\left(\frac{x}{y}\right)^3 - 2\right| = \left|\frac{x}{y} - \sqrt[3]{2}\right| \left|\frac{x}{y} - \sqrt[3]{2}e^{2\pi i/3}\right| \left|\frac{x}{y} - \sqrt[3]{2}e^{-2\pi i/3}\right| > \frac{c}{y^{5/2}} \left(\frac{13}{2}\right)^2$$

$$z^3 - 2 = (z - \sqrt[3]{2})(z - \sqrt[3]{2}e^{2\pi i/3})(z - \sqrt[3]{2}e^{-2\pi i/3})$$

$$\frac{1}{y^3} > \frac{c}{y^{5/2}} \Leftrightarrow y^{1/2} < \frac{1}{c} \quad |y| < c''$$

So x, y have only finitely many solutions.

(Alan Baker "linear forms in logarithms")

Prop:

$\alpha \notin \mathbb{Q}$ then $\sum_{h=1}^N \min\left(N, \frac{1}{\|h\alpha\|}\right) = o(N^2)$

($\Rightarrow \{\alpha n^2\}$ is UD mod 1, $\forall \alpha \notin \mathbb{Q}$)

Proof:

Use Dirichlet's lemma to find coprime a, q , $1 \leq q \leq N$, $\left|\alpha - \frac{a}{q}\right| \leq \frac{1}{qN}$

i.e. $\alpha = \frac{a}{q} + \frac{\theta}{N}$, $|\theta| \leq 1$

$$\|h\alpha\| = \left\| \frac{ha}{q} + \frac{\theta h}{N} \right\|$$

"
dist($h\alpha, \mathbb{Z}$)

Note: $h \rightarrow h + kq$ then you haven't changed this expression by much.

Write $(M-1)q < N \leq Mq$

$$\text{Our Sum} \leq \sum_{h=1}^{Mq} \min(N, \|h\delta\|) \leq \sum_{k=1}^{M-1} \max_{|\tilde{\theta}| < 1} \sum_{j=1}^q \min(N, \|\frac{a_j}{2} + \frac{\tilde{\theta}}{2}\|)$$

$h = kq + j$
 $j = 1, 2, \dots, q$
 $0 \leq k < M \leq N/q$

$$= M \cdot \max_{|\tilde{\theta}| < 1} \sum_{j=1}^q \min(N, \|\frac{a_j}{2} + \frac{\tilde{\theta}}{2}\|) =$$

change variables: As j runs over $\mathbb{Z}/q\mathbb{Z}$, so does aj
 Since $(a, q) = 1 \dots aj \mapsto i$

$$= M \cdot \max_{|\tilde{\theta}| < 1} \sum_{i=1}^q \min(N, \|\frac{i}{q} + \frac{\tilde{\theta}}{2}\|)$$

There is at most ONE value of $\tilde{i}$ such that $\|\frac{\tilde{i} + \tilde{\theta}}{2}\| < \frac{1}{2q}$
 for this $\tilde{i}$, take N in minimum. For other i 's

$$\|\frac{i}{q} + \frac{\tilde{\theta}}{2}\| > \frac{1}{2} \min(\frac{i}{q}, \frac{q-i}{q})$$

$$\text{So our Sum} \leq \frac{N}{2} \cdot \left(N + 2 \sum_{i=1}^{q/2} \frac{1}{i/q} \right) \ll$$

$$\ll \frac{N^2}{2} + \frac{N}{2} q \cdot \sum_{i=1}^{q/2} \frac{1}{i} \ll \frac{N^2}{2} + N \log N$$

$\ll \log q \leq \log N$
 $O(N^2)$

Now recall $q \rightarrow \infty$ as $N \rightarrow \infty$ since $\delta \notin \mathbb{Q}$ so $\frac{N^2}{2} = o(N^2)$

Thm. (Weyl, 1916):

$P(x) = a_d x^d + a_{d-1} x^{d-1} + \dots + a_1 x + a_0$ polynomial s.t. at least one of $a_d, a_{d-1}, \dots, a_1$ (not including the a_0) is irrational
 Then the sequence $\{P(n) \mid n=1, 2, \dots\}$ is UD mod 1.

See book Kuipers & Niederreiter for streamlined argument due to Vander Corput.