

Topics: "open problems in number theory"

1) The golden Ratio. "normal"

$$\varphi = \frac{1+\sqrt{5}}{2} = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{\ddots}}} = \lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n}, \quad F_{n+1} = F_n + F_{n-1} \text{ Fibonacci sequence}$$

Def:

A number $\alpha = \dots a_1 a_2 \dots a_n \dots$ is simply normal (in base 10) if every digit $(0, 1, 2, \dots, 9)$ appears with same frequency $= 1/10$

$$\lim_{N \rightarrow \infty} \frac{1}{N} \# \{n \leq N \mid a_n = t\} = 1/10, \text{ ditto for any digit } t = 0, 1, 2, \dots, 9$$

Def:

A real number $\alpha = \dots a_1 a_2 \dots a_n \dots$ is Normal (to base 10) if

$\forall k \geq 1$, every string of k digits (eg $k=3, 127$)

appears with the same frequency $= 1/10^k$

ex:

Find a simply normal number which is NOT normal (to base 2)

Question: $\swarrow$ real algebraic, $\pi, e, \dots$

Is any "naturally occurring" number Normal (to base 10)

non-natural example (Champernowne 1932)

$0.123456789101112131415\dots$ is normal to base 10.

2) "The circle problem" (Hardy 1920's)

count lattice pts in a ball

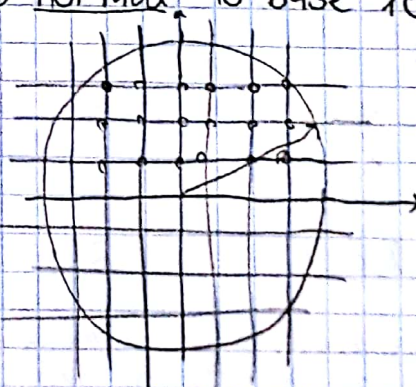
$$N(R) = \# \{n = (n_1, n_2) \in \mathbb{Z}^2 \mid n_1^2 + n_2^2 \leq R^2\}$$

$$\text{Easy: } N(R) \sim \text{area } B(0, R) = \pi R^2$$

$$P(R) := N(R) - \pi R^2$$

$$P(R) = o(R^2) \quad \frac{P(R^2)}{R^2} \longrightarrow 0$$

little oh



Gauss: $p(R) \ll R \Rightarrow$ length of circumference

Hardy's conjecture:

$$p(R) \ll \left(\sqrt{\text{length of circumference}} \right)^{1+\varepsilon} \quad \forall \varepsilon > 0, \ll R^{1/2+\varepsilon}$$

Sierpinski; Van der Corput 1906: $p(R) \ll R^{2/3}$

3) Let $r(n) = \#$ of lattice pts on the circle $x^2 + y^2 = n$

$$\text{So } \sum_{n \leq R^2} r(n) = N(R) \sim \pi R^2$$

so "on average" over all $n \leq R^2$ $r(n)$ is π . But in fact for "almost all" n 's, $r(n) = 0$.

E. Landau (1908):

$$\# \{ n \leq N \mid n = \square + \square \} \sim \kappa \frac{N}{\sqrt{\log N}}$$

$\Rightarrow$ The "average gap" between consecutive integers $n = \square + \square \leq N$ is $\asymp \sqrt{\log N}$

Question:

What is the maximal gap?

Let order: $n_1 = 1 = 1^2 + 0^2$, $n_2 = 2 = 1^2 + 1^2$, $n_3 = 4 = 2^2 + 0^2$, $n_4 = 5 = 2^2 + 1^2$...

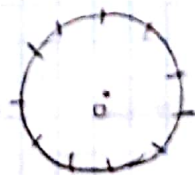
$n_i = i^{\text{th}}$ sum of 2 squares

Conj.:

$$G(x) = \max_{n \leq x} (n_{i+1} - n_i) \ll x^\varepsilon, \quad \forall \varepsilon > 0$$

Folklore: $G(x) \leq x^{1/4}$

4) Lattice points in short arcs.



Take a circle of radius $R = \sqrt{n}$, have $r(n) > 0$ lattice pts on it.

Can we have close lattice pts? Total number is $r(n)$.

length of circle is $2\pi R = 2\pi \sqrt{n}$

$\Rightarrow$ average distance is $\frac{2\pi n}{r(n)} \gg n^{1/2} - \epsilon$

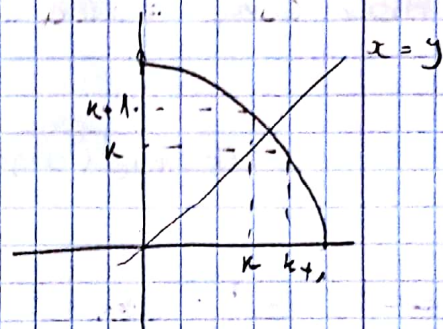
Know:

$$r(n) \ll n^\epsilon \quad \forall \epsilon > 0.$$

Example:

$$n = k^2 + (k+1)^2 \quad A = (k, k+1) \\ B = (k+1, k)$$

$$\text{dist}(A, B) = \sqrt{2} = O(1)$$



Jarnik (1940?):

An arc of length $< R^{1/3}$ cannot contain 3 lattice points.

Conj.:

Fix $\delta < 1$. Then an arc of length R^δ contains at most $M(\delta) < \infty$ lattice points.

Jarnik:

$$\text{OK } \delta = 1/3, \quad M(1/3) = 3$$

Cilleruelo & Cordoba (1992):

$$\text{OK } \delta < \frac{1}{2}$$

$$\boxed{?? : \delta = 1/2, ??}$$

5) The ABC conjecture (Masser & Oesterlé 1985)

A, B, C coprime integers s.t. $A+B=C$

$$\forall \epsilon > 0, \exists c_{\epsilon} > 0 \text{ max } (|A|, |B|, |C|) < c_{\epsilon} \text{rad}(ABC)^{1+\epsilon}$$

$$\text{Where: } \text{rad}(\prod p_i^{l_i}) = \prod p_i$$

$\Rightarrow$ An asymptotic version of Fermat's last theorem Wiles (1994)
($n \geq 3$ $x^n + y^n = z^n$ only has trivial solutions)

pf (ABC $\Rightarrow$ Fermat):

$A = x^n, B = y^n, C = z^n$, x, y, z coprime.

ABC conj $\Rightarrow \forall \epsilon > 0, C = z^n < c(\epsilon) \text{rad}(x^n y^n z^n)^{1+\epsilon}$

$$= c(\epsilon) \text{rad}(xyz)^{1+\epsilon} \leq c(\epsilon) (xyz)^{1+\epsilon} \leq c(\epsilon) z^{3+3\epsilon}$$

$$\text{Rad}(N) \leq N$$

$$\Rightarrow z^n < c(\epsilon) z^{3+\epsilon} \mapsto z^{n-3-\epsilon} < c(\epsilon)$$

$$\epsilon = \frac{1}{2} \Rightarrow \exists c(\frac{1}{2}) = c > 0 \text{ s.t. } \forall x, y, z \quad z^{n/2} \leq z^{n-3-\frac{1}{2}} < c$$

$$\Rightarrow z < c^2$$

$\Rightarrow \forall n \geq 4, \exists$ only finitely many solutions to $x^n + y^n = z^n$
(Proved by Faltings 1984)

Normal Numbers

Def:

$\alpha = 0.a_1 a_2 \dots \in (0, 1)$ is normal to base 10 if $\forall k \geq 1$,

$\forall$ string $t_1 \dots t_k, t_i \in \{0, \dots, 9\}$

$$\frac{1}{N} \# \{ n \leq N \mid a_{n+1} = t_1, a_{n+2} = t_2, \dots, a_{n+k} = t_k \} \xrightarrow{N \rightarrow \infty} \frac{1}{10^k}$$

Ex: $\frac{1}{2}$ is not normal $\frac{1}{2} = 0.500\dots = 0.4999\dots$

Exercise:

Rationals are never normal.

Un-Natural Examples:

Champernowne 1952 : $0.123456789101112\dots$ normal to base 10.

Copeland & Erdos 1946, Conj. by Champernowne:

$0.23571113171923\dots$ normal to base 10.

Davenport & Erdos 1952 (Conj. Copeland & Erdos):

take $f(x) \in \mathbb{Z}[X]$ s.t $f(x) > 0, x=1, 2, 3, \dots$, take $\alpha = 0.f(1)f(2)f(3)\dots$
is normal to base 10.

Conj. (E. Borel 1950):

real algebraic $\alpha \notin \mathbb{Q}$ is normal to any base.

Problem:

Find a "natural" example of a normal number.

Conj: π is normal

Uniform Distribution

Let $X = \{x_1, x_2, x_3, \dots\} \subset \mathbb{R}/\mathbb{Z} \cong [0, 1)$ be a sequence of points on the circle.

We say X is uniformly distributed if $\forall$ interval $I = [a, b] \subset \mathbb{R}/\mathbb{Z}$

$$\lim_{N \rightarrow \infty} \frac{1}{N} \# \{n \leq N \mid x_n \in I\} = \text{length } I$$

Note:

X is U.D. $\Rightarrow X$ is dense in $\mathbb{R}/\mathbb{Z}$

Generally, $X = \{x_n\} \subset \mathbb{R}$ then look at fractional parts

$x = \text{integer} + \{x\}$, $0 \leq \{x\} \leq 1$. then X is U.D mod 1 if the fractional parts are U.D.

Ex:-

$x_n = \{\log n\}$, $n = 1, 2, 3, \dots$ is dense in $[0, 1)$. we
fractional part.

will see that it is not Uniformly distributed.

Example: (Stern)

$\alpha \in \mathbb{R} \setminus \mathbb{Q}$ irrational, $x_n = \{\alpha \cdot n\}$, $n = 1, 2, 3, \dots$ fractional part

Claim:

This is uniformly distributed $\Leftrightarrow$ dense.

Proposition:

α normal to base 10 $\iff \{\alpha \cdot 10^n\}$ is u.d.

Proof:

Assume α is normal. want u.d. Take interval $I \subset [0,1)$ & show $\frac{1}{N} \#\{n \leq N \mid \{\alpha \cdot 10^n\} \in I\} \xrightarrow{N \rightarrow \infty} \text{length}(I)$

check, WLOG: $I = [0.a_1 a_2 \dots a_k, 0.a_1 \dots (a_k+1))$

$$\#\{n \leq N \mid \{\alpha \cdot 10^n\} \in [0.123, 0.124)\}$$

$$\alpha = 0.537189327$$

$$10\alpha = 5.37189327 \dots \quad \{10\alpha\} = 0.37189327 \dots$$

$$\{10^2\alpha\} = 0.7189327 \dots$$

$$\{10^3\alpha\} = 0.189327 \dots$$

$$\{10^n\alpha\} = 0.a_{n+1}a_{n+2}\dots \quad \alpha = 0.a_1a_2\dots$$

$$\{10^n\alpha\} \in I = [0.123, 0.124) \iff a_{n+1}=1, a_{n+2}=2, a_{n+3}=3$$

$$\frac{1}{N} \#\{n \leq N \mid \{\alpha \cdot 10^n\} \in I\} = \frac{1}{N} \#\{n \leq N \mid a_{n+1}=1, a_{n+2}=2, a_{n+3}=3\}$$

$$\text{So u.d.: LHS} \longrightarrow \text{length}(I) = 0.001 = \frac{1}{10^3}$$

$$\text{Normality: RHS} \longrightarrow \frac{1}{10^3} \text{ so we get equivalence}$$

H. Weyl (1916): Weyl's Criterion

$\chi = \{x_n \mid n=1,2,3,\dots\} \subseteq \mathbb{R}$ is u.d mod 1 $\iff$

$$\forall \text{ integer } k \neq 0 \quad \frac{1}{N} \sum_{n=1}^N e^{2\pi i k x_n} \xrightarrow{N \rightarrow \infty} 0$$

Thm:

If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ irrational, then $\{\alpha n \mid n=1,2,\dots\}$ is u.d mod 1.

$$\text{Look at Weyl sums } S_N(N) = \frac{1}{N} \sum_{n=1}^N e^{2\pi i k \alpha n} =$$

$$= \frac{1}{N} \sum_{n=1}^N (e^{2\pi i k \alpha})^n =$$

$$= \begin{cases} 1 & , e^{2\pi i k \alpha} = 1 \text{ cannot happen if } \alpha \notin \mathbb{Q} \\ \frac{1}{N} \frac{(e^{2\pi i k \alpha}) - (e^{2\pi i k \alpha})^{N+1}}{1 - e^{2\pi i k \alpha}} \end{cases}$$

$$|S_N(N)| \leq \frac{1}{N} \left| \frac{2}{1 - e^{2\pi i k \alpha}} \right| \xrightarrow{N \rightarrow \infty} 0$$

Example: That can be done using Weyl's sums: $\alpha n^2, \alpha \notin \mathbb{Q}$

Example:

The sequence of fractional parts of $\log n$ is not uniformly distributed

By Weyl's criterion, it suffices to show:

$$\frac{1}{N} \sum_{n=1}^N e^{2\pi i \log n} \xrightarrow{N \rightarrow \infty} 0$$

Summation By Parts: (Abel's Identity)

Let $\{a(n)\}_{n=1}^{\infty}$ be a sequence, $f(t)$ differentiable.

$$A(x) := \sum_{1 \leq n \leq x} a(n) \quad (A(x) = 0, x < 1)$$

$$\text{Then } \sum_{1 \leq n \leq x} a(n) f(n) = A(x)f(x) - \int_1^x A(t)f'(t) dt$$

To compare: Integration by parts:

$$\int_1^x A'(t)f(t) dt = A(t)f(t) \Big|_1^x - \int_1^x A(t)f'(t) dt$$

Apply to $f(t) = e^{2\pi i \log t}$ $a(n) = 1$.

$$\sum_{n=1}^N e^{2\pi i \log n} = \sum_{1 \leq n \leq N} a(n)f(n) \quad \textcircled{=}$$

$$f(t) = t^{2\pi i}, \quad f'(t) = 2\pi i t^{2\pi i - 1}, \quad A(t) = \sum_{1 \leq n \leq t} a(n) = \sum_{1 \leq n \leq t} 1 = \lfloor t \rfloor$$

$$\textcircled{=} A(N)f(N) - \int_1^N A(t)f'(t) dt = N N^{2\pi i} - \int_1^N \lfloor t \rfloor 2\pi i t^{2\pi i - 1} dt = N + \lfloor N \rfloor + f(N)$$

$$= N N^{2\pi i} - 2\pi i \int_1^N t t^{2\pi i - 1} dt + 2\pi i \int_1^N \{t\} t^{2\pi i - 1} dt =$$

$$= N N^{2\pi i} - 2\pi i \frac{t^{2\pi i + 1}}{2\pi i + 1} \Big|_1^N + O\left(\int_1^N \frac{1}{t} dt\right) =$$

$$= N N^{2\pi i} - 2\pi i \frac{N^{1+2\pi i}}{1+2\pi i} + O(1) + O(\log N)$$

$$\Rightarrow \frac{1}{N} \sum_{n=1}^N e^{2\pi i \log n} = \frac{N^{2\pi i}}{1+2\pi i} + O\left(\frac{\log N}{N}\right) \xrightarrow{N \rightarrow \infty} 0$$

Proof of Summation by parts: (assume x - integer)

$$\sum_{n=1}^x a(n)f(n) = \sum_{n=1}^x \{A(n) - A(n-1)\} f(n) =$$

$$a(n) = A(n) - A(n-1)$$

$$= \sum_{n=1}^x A(n)f(n) - \sum_{n=1}^x A(n-1)f(n) = \sum_{n=1}^x A(n)f(n) - \sum_{m=0}^{x-1} A(m)f(m+1) =$$

$$= A(x)f(x) - A(0)f(1) - \sum_{n=1}^{x-1} A(n)\{f(n+1) - f(n)\} =$$

$$= A(x)f(x) - \sum_{n=1}^{x-1} A(n) \int_n^{n+1} f'(t) dt =$$

$$= A(x)f(x) - \sum_{n=1}^{x-1} \int_n^{n+1} A(n) f'(t) dt = A(x)f(x) - \sum_{n=1}^{x-1} \int_n^{n+1} A(t) f'(t) dt =$$

$$= A(x)f(x) - \int_1^x A(t)f'(t) dt.$$