

Random matrices - lecture #1

24/10/2017

Theme:

$$A = \underbrace{\left(\text{matrix} \right)}_N \left\{ \begin{array}{l} N \\ A \text{ is random} \end{array} \right.$$

Study the eigenvalues & eigenvectors of A .

Type of theorems we'll see:

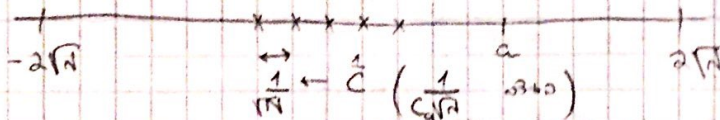
Random matrix A will be Hermitian, $A = A^*$

$\Rightarrow$ eigenvalues are real.

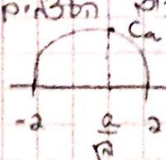
Wigners model:

$A = \left(\begin{array}{c} \text{matrix} \\ N \end{array} \right)$ the entries are Independent, identically dist. except for being Herm (meaning the upper triangle is distributed as mentioned)

particularly important: Gaussian entries, $N(0,1)$

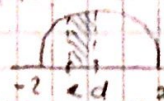


1)



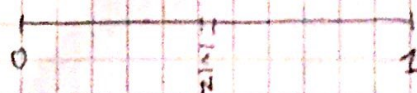
this is Wigners semi-circle distributed.

problem 1300: find the probability that the eigenvalues of a random matrix are in the interval (c,d) .



Wigner de

another Problem:



N uniform independent points

$$P\left(\text{to see 2 points in an interval of length } \frac{1}{N}\right) \leq C^2$$

2) however, Eigenvalues have repulsion, same Prob. will be $e^{-\alpha}$, $\alpha > 2$

these will lead us to: $\Rightarrow$ Determinantal Process, Sine Kernel

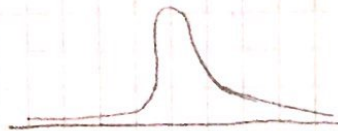
3) Maximal eigenvalues.

λ - max eigenvalue

$$E(\lambda) \sim 2\sqrt{N}$$

$$\text{std}(\lambda) \sim cN^{1/6}$$

$$\frac{\lambda - E(\lambda)}{\text{std}(\lambda)} \xrightarrow{d} \text{Tracy - Widom distribution}$$



Gaussian distribution:



a mathematical application:

Longest increasing subsequence.

Permutation $\pi \in S_N$ chosen uniformly.

for example: $N=7$: $3 \ 5 \ 1 \ 6 \ 2 \ 7 \ 4$ $z(\pi) = 4$

$\therefore$ these are increasing subsequences

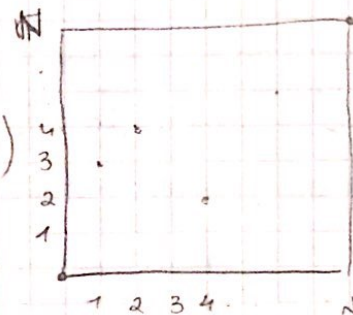
$z(\pi)$ - length of the longest increasing subsequence

Ulam 1960:

$$E(z(\pi)) \sim 2\sqrt{N} \quad (\text{Vershik - Kerov})$$

$$\text{std}(z(\pi)) \sim cN^{1/6} \quad (\text{Benf. Deift - Johanson})$$

$$\frac{z(\pi) - E(z(\pi))}{\text{std}(z(\pi))} \xrightarrow{d} \text{Tracy widom}$$

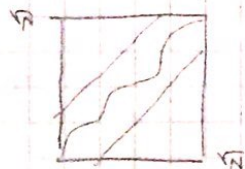


the longest increasing subsequence is equivalent to finding the longest path from $(0,0)$ to (n,n) (as in the path going through as many points as possible).

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(הנקודות) $\sqrt{N}$ אורכו

N uniform points



Random operators:

$$\begin{pmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & 1 & -2 & 1 \\ & & 1 & -2 & 1 \\ & & & 1 & -2 & 1 \\ & & & & \ddots & \ddots \\ & & & & & 1 & -2 & 1 \\ & & & & & & 1 & -2 & 1 \\ & & & & & & & 1 & -2 & 1 \\ & & & & & & & & 1 & -2 & 1 \end{pmatrix} - \text{Laplacian}$$

this matrix corresponds to the probability of moving to one of the adjacent cells.

Anderson Localization:

$$\begin{pmatrix} V_1 & 1 & & & \\ 1 & V_2 & 1 & & \\ & 1 & V_3 & 1 & \\ & & & \ddots & \ddots \\ & & & & 1 & V_n \end{pmatrix} \quad V_i \text{ are IID } N(0,1)$$

the question is how do the random variables effect the movement of the particle? turns out that the Eigenvectors where the particle is" in physics you want to study the position of the particle - will it be localized (stuck) or will its position change - delocalized.

$$\begin{pmatrix} \text{diagonal} & & \\ & \text{diagonal} & \\ & & \ddots \\ & & & \text{diagonal} \end{pmatrix}$$

Here. there are W diagonals of IID, $N(0,1)$ random variable. this is the Band Matrix.
when $W \ll n$ the position of the particle is localized.

Book: Anderson, Guionnet, Zeitouni, an introduction to random matrices.

Grade: based on homeworks.

Wigner model:

Two families of ^{real} IID random Var. $(Z_{i,j})_{i,j=1}^N$, $(Y_i)_{i=1}^N$
 IID ↑ independent ↑ IID

Random matrix X_N , $N \times N$ matrix.

$$X_N = \begin{pmatrix} Y_1 & & \\ & \ddots & \\ & & Y_N \\ & & & Z_{i,j} \\ & & & & \ddots \\ & & & & & Y_N \end{pmatrix} \cdot \frac{1}{\sqrt{N}} \quad \text{Symmetric}$$

here we scale the eigenvalues of this matrix.

Assumptions:

- 1) $\max(E(|Z_{i,j}|^k), E(|Y_i|^k)) < \infty : k \geq 8$
- 2) $E(Z_{i,j}) = 0$, $\text{Var}(Z_{i,j}) = 1$
- 3) $E(Y_i) = 0$.

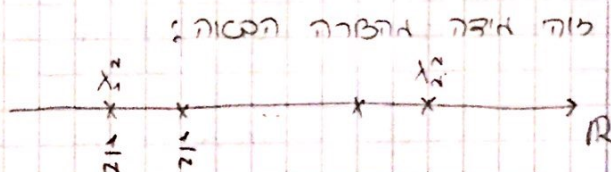
we study the empirical eigenvalue dist:

$\lambda_1^N \leq \dots \leq \lambda_N^N$ eigenvalues.

$$Z_N = \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i^N}$$

$f: \mathbb{R} \rightarrow \mathbb{R}$ function:

$$\langle Z_N, f \rangle = \int f(x) dZ_N(x) = \frac{1}{N} \sum_{i=1}^N f(\lambda_i^N)$$



Semi-circle: measure on $\mathbb{R}$ given by: $d\sigma = \frac{1}{2\pi} \sqrt{4-x^2} \mathbb{1}_{[-2,2]} dx$

$$\langle \sigma, f \rangle = \int f(x) d\sigma(x) = \int_{-2}^2 f(x) \cdot \frac{1}{2\pi} \sqrt{4-x^2} dx$$

we will want to prove that the Z_N measure converges to σ

Reminder:

A sequence of probability measures (μ_n) on $\mathbb{R}$ converges to a probability measure ν on $\mathbb{R}$ (weak convergence, in distribution) if for every bounded continuous function f on $\mathbb{R}$:

$$\int f(x) d\mu_n(x) \xrightarrow{n \rightarrow \infty} \int f(x) d\nu(x)$$

there is a slight twist here for Z_n is a Random measure.

Theorem, (Wigner):

Z_n converges to σ weakly in probability in the sense that:

$\forall f$, bounded, continuous & $\forall \epsilon > 0$,

$$P\left(\left|\int f(x) dZ_n(x) - \int f(x) d\sigma(x)\right| > \epsilon\right) \xrightarrow{n \rightarrow \infty} 0$$

Basic tool: $\text{tr}(X_n) = \sum_{i=1}^N \lambda_i^N$

$$\text{tr}(X_n^k) = \sum_{i=1}^N (\lambda_i^N)^k$$

Method of moments:

(μ_n) prob. measures, ν prob. measures.

given: $\int x^k d\mu_n(x) \xrightarrow{n \rightarrow \infty} \int x^k d\nu(x) \quad \forall k \geq 1 \text{ int.}$

Mausdorff: μ of bounded support implies $\mu_n \xrightarrow{d} \nu$

general ν : Not necessarily, but yes if $\int x^k d\nu$ grows "not too fast" with k .

Moments of semi-circle law

$$m_k := \langle \sigma, x^k \rangle = \int x^k d\sigma(x) \quad \forall k \geq 0 \text{ int.}$$

Proposition:

Catalan number

$$m_k = 0 \text{ if } k \text{ is odd. } m_{2k} = C_k = \frac{1}{k+2} \binom{2k}{k} = \frac{2k!}{k!(k+1)!}$$

Proof:

$m_k = 0$ for k odd, by symmetry of σ .

for $k \geq 1$, establish a recursion:

$$\begin{aligned} m_{2k} &= \int_{-2}^2 x^{2k} \frac{1}{2\pi} \sqrt{4-x^2} dx = \int_{-\pi/2}^{\pi/2} 2^k \frac{2^k}{2\pi} \sin^{2k}(\theta) \cos^2(\theta) d\theta = \\ &= \frac{2}{\pi} \cdot 2^{2k} \left[\int_{-\pi/2}^{\pi/2} \sin^{2k}(\theta) d\theta - \int_{-\pi/2}^{\pi/2} \sin^{2k+2}(\theta) d\theta \right] = (*) \end{aligned}$$

$x = 2\sin\theta$
 $-\pi/2 \leq \theta \leq \pi/2$

$\cos^2\theta = 1 - \sin^2\theta$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2k+2}(\theta) d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \overset{\text{diff}}{\sin^{2k+1}(\theta)} \overset{\text{int}}{\sin(\theta)} d\theta =$$

$$= -(2k+1) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2k}(\theta) \cos^2(\theta) d\theta = \frac{\pi}{2 \cdot 2^{2k}} (2k+1) m_{2k}$$

$$\Rightarrow (*) = \frac{2}{\pi} 2^{2k} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2k}(\theta) - (2k+1) m_{2k}$$

$$\Rightarrow m_{2k} = \frac{2}{\pi(2k+2)} 2^{2k} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2k}(\theta) \overset{\text{as before}}{=} \frac{4(2k-1)}{2k+2} m_{2k-2}$$

have also $m_0 = 1$. Check that Catalan numbers satisfy this.