WMSD3
25/11/14

Random Spanning Trees

Ex. 2 now online.

~~Class in Hebrew today (?!?!)~~

Random spanning forests

G - random, infinite connected weighted graph (even infinite - countable - degrees allowed as long as $\forall x \sum_y c_{xy} < \infty$, "locally finite").

Let G_n be a finite exhaustion.

Put μ_n^F for the UST measure on G_n.

For any finite set of edges $B = \{e_1, \dots, e_m\}$ $\exists N$ $\forall n > N$, $B \subseteq G_n$, so $\mu_n^F(B \subseteq UST) = \prod_{k=1}^{m} \mu_n^F(e_k \in T \mid e_1, \dots, e_{k-1} \in T) =$

$\prod_{k=1}^{m} \mu_n^F(e_k \in T; G_n/\{e_1, \dots e_{k-1}\}) \overset{\text{Kirchhoff}}{=} \prod_{k=1}^{m} I(e_k; G_n/\{e_1 \dots e_{k-1}\}) \overset{\text{R. monotonicity}}{\geq}$

$\prod_{k=1}^{m} I(e_k; G_{n+1}/\{e_1, \dots, e_{k-1}\}) = \mu_{n+1}^F(B \subseteq UST)$. So $\lim_{n\to\infty} \mu_n^F(B \subseteq UST)$ converges. Call the limit $\mu^F(B)$. Similarly

$\mu_n^F(B \subseteq UST, A \cap UST = \phi)$ converges (using inclusion-exclusion $\mu\left(\begin{smallmatrix} e_1 \in UST \\ e_2 \notin UST \end{smallmatrix}\right) = \mu(e_1 \in UST) - \mu(e_1, e_2 \in UST)$.

Now, Kolmogorov extension thm.: suppose ν_k are prob. measures on $\mathbb{R}^k$ that are consistent, i.e. $\nu_{k+1}([a_1,b_1] \times \dots \times [a_k,b_k] \times \mathbb{R}) = \nu_k([a_1,b_1] \times \dots \times [a_k,b_k])$ then $\exists$ prob. measure on $\mathbb{R}^{\mathbb{N}}$ ν s.t. $\nu(\{w \mid w_i \in [a_i,b_i]\ 1 \leq i \leq k\}) = \nu_k(-)$. So $\exists$ prob. measure on subgraphs of G (2^E) - the Free Uniform Spanning Tree, FUSF / FSF.

Wired version Let $G_n^W = \left(\begin{smallmatrix}\text{identify } G \setminus G_n \\ \text{to a single vertex } z_n\end{smallmatrix}\right)$.

$\mu_n^W = UST$ on G_n^W. By a similar argumet (increasing instead of decreasing) $\mu_n^W(B) \leq \mu_{n+1}^W(B)$ and we can describe μ^W prob. measure on (2^E) subgraphs of G - "Wired Uniform Spanning Forest", WUSF/WSF.

Props: μ^F and μ^W don't depend on choice of exhaustion. By R. monotonicity – can always interlace.

$\forall e_1, \dots e_m$ a cycle, $\mu^{F/W}(e_1, \dots e_m) = 0$ (since they are contained in all G_n from a certain point onwards).

also $\mu^{F\backslash W}$ ~~–~~ a.s. spanning by the same argument.

$\mu^{F/W}$ are invariant to graph automorphisms – if $\varphi: G \to G$ is a graph automorphism (network auto. too – conserves weights). Then any event A measurable in the sigma-algebra generated by cylinders, $\mu^{F/W}(A) = \mu^{F/W}(\varphi A)$ (consider $\varphi G_n \dots$).

G recurrent $\to \mu^F = \mu^W$, since the probability that a random walker on G will reach $G \backslash G_n$ tends to 0 as $n \to \infty$.

Example G = 3-reg. infinite tree. $\mu^F(T = G) = 1$. We claim $\mu^W(e \in T) < 1$.

$P^{G_n^W}(\tau_{e_+} < \tau^+_{e^-}) \geq \frac{1}{3} + c$ so $\mu^W(e \in T) < 1$.

μ^F, μ^W – a.s. all trees are infinite.

$I(e; G_n) \geq I(e; G_n^W)$, so $\mu^F(e \in T) \geq \mu^W(e \in T)$.

Also, by cor. from last week, for any increasing event A, $\mu^F(A) \geq \mu^W(A)$. This implies you can couple μ^F and μ^W s.t. WUSF $\subseteq$ FUSF, i.e. $\exists$ prob. measure P on pairs of spanning forests s.t. $P((A, 2^E)) = \mu^W(A)$, $P((2^E, A)) = \mu^F(A)$ but $P(\{(A_1, A_2) \mid A_1 \subseteq A_2\}) = 1$.

$A \subseteq 2^{E(G)}$

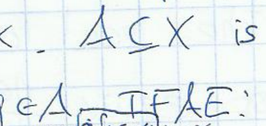
Coupling – example $p < q$, product measures on $\{0,1\}^n$.

To couple, let $u_i \sim \mathrm{unif}[0,1]$, $x = (\mathbb{1}_{(u_i \leq p)})_{1 \leq i \leq n}$ and $y = (\mathbb{1}_{(u_i < q)})_{1 \leq i \leq n}$, so a.s. $x \leq y$ pointwise, $x \sim p$ iid prod., $y \sim q$ iid prod. ...

Thm. (Strassen's) Let $(X, \leq)$ be a partially ordered set, μ_1 & μ_2 two prob. measures on X. $A \subseteq X$ is increasing iff $x \in A$ & $y \geq x \to y \in A$. TFAE:

i) $\forall A$ increasing, $\mu_1(A) \leq \mu_2(A)$.

ii) $\exists$ coupling of μ_1, μ_2.



2 $\to$ 1. Immediate.

1 $\to$ 2. Max-flow min-cut.