W4 D4
19/11/14

Random Spanning Trees

A subset $A \subset 2^E$ is <u>increasing</u> iff $\forall B \in A, \forall e \in E,$ $B \cup \{e\} \in A$.

We say A ignores $e \in E$ iff $\forall B \in A, B \cup \{e\}, B \setminus \{e\} \in A$.

<u>Thm.</u> G conn., finite, weighted. P UST measure.
If A increasing ignores e, $P(A \mid e \in T) \leq P(A)$.

<u>Pf.</u> By induction on the number of edges. When $|V|=2$ this is true: $P(A \mid e \in T) > 0 \longleftrightarrow A = 2^E$, since A ignores e and if $e \in T$ no other edge is in T.

Assume $|V| \geq 3$. In G/e each spanning tree has $|V|-2$ edges, so $\sum_{f \in E \setminus \{e\}} P(A, f \in T \mid e \in T) = (|V|-2) P(A \mid e \in T)$.

But this is $P(A \mid T) \sum_{f \in E \setminus \{e\}} P(f \in T \mid e \in T)$. Assume the sum isn't 0 - o/w we're fine. So there must be an element in the sums $f \in E \setminus \{e\}$ with $P(f \in T \mid e \in T) > 0$ and $P(A, f \in T \mid e \in T) \geq P(A \mid e \in T) P(f \in T \mid e \in T)$ (since $P(A, f \in T \mid e \in T) = 0$ if $P(f \in T \mid e \in T) = 0$).

So, $P\left(A \mid \begin{smallmatrix} f \in T \\ e \in T \end{smallmatrix}\right) \geq P(A \mid e \in T)$, which implies

$P\left(A \mid \begin{smallmatrix} f \in T \\ e \in T \end{smallmatrix}\right) \geq P\left(A \mid \begin{smallmatrix} e \in T \\ f \notin T \end{smallmatrix}\right)$. Now,

$P(A \mid e \in T) = P(f \in T \mid e \in T) P\left(A \mid \begin{smallmatrix} e \in T \\ f \in T \end{smallmatrix}\right) + P(f \notin T \mid e \in T) P\left(A \mid \begin{smallmatrix} e \in T \\ f \notin T \end{smallmatrix}\right) \leq$

since $P(f \in T \mid e \in T) \leq P(f \in T)$

$\leq P(f \in T) P\left(A \mid \begin{smallmatrix} e \in T \\ f \in T \end{smallmatrix}\right) + P(f \notin T) P\left(A \mid \begin{smallmatrix} e \in T \\ f \notin T \end{smallmatrix}\right)$. By induction,

$P\left(A \mid \begin{smallmatrix} e \in T \\ f \in T \end{smallmatrix}\right) \leq P(A \mid f \in T)$ (This is just $P(A/f \mid e \in T)$ vs. $P(A/f)$ where $A/f = \{F \subseteq 2^{E \setminus \{f\}} \mid F \cup \{f\} \in A\}$).

Also by induction $P\left(A \mid \begin{smallmatrix} e \in T \\ f \notin T \end{smallmatrix}\right) \leq P(A \mid f \notin T)$. (Same, for $P(A \setminus f \mid e \in T)$ vs. $P(A \setminus f)$ where $A \setminus f = \left\{ \begin{smallmatrix} F \subseteq 2^{E \setminus \{f\}} \\ F \in A \end{smallmatrix} \right\}$).

So, $P(A \mid e \in T) \leq P(f \in T) P(A \mid f \in T) + P(f \notin T) P(A \mid f \notin T) =$
$P(A)$ [conditioned on $f \in T$ or $f \notin T$]. WWT.

Cor. Let G be a conn. subgraph of a finite conn graph H with μ_G, μ_H UST measures. Then $\mu_G(A) \geq \mu_H(A)$ (for any increasing $A \subseteq 2^{E(G)}$).

In the subgraph G

Pf. By induction, it suffices to show for H with one edge e more then G. $\mu_H(A) = \mu_H(e \in T)\mu_H(A | e \in T) + \mu_H(e \notin T)\mu_H(A | e \notin T)$. If $\mu_H(e \notin T) = 0$ then $\mu_H(A) = \mu_G(A)$.
o\w, by prev. thm., $\mu_H(A | e \in T) \leq \underbrace{\mu_H(A) \leq \mu_H(A | e \notin T)}$ $= \mu_G(A)$

so $\mu_H(A) \leq \mu_H(e \in T)\mu_G(A) + \mu_H(e \notin T)\mu_G(A) = \mu_G(A)$.

Actually, this proves it.