W4 D3

18/11/14

Random Spanning Trees

Reminder $\ell^2_-(E, \{r_e\})$ with a basis $\chi^e = 1_{\vec{e}} - 1_{\overleftarrow{e}}$, orthogonal but not necessarily of norm 1.

$\langle\theta, \theta'\rangle = \frac{1}{2}\sum_{e\in\vec{E}} r_e\theta(e)\theta'(e) = \sum_{e\in E_{1/2}} r_e\theta(e)\theta'(e)$

a star at v is $\sum_{e^+=v} \chi^e$. STAR = span(star(v)).

Given a directed cycle $e_1, \dots e_n$ in G the $\ell^2_-(E)$ element $\sum_{i=1}^n \chi^{e_i}$ is the cycle in $\ell^2_-(E)$. We define CYCLE = span(cycles).

In finite G we observed STAR $\oplus$ CYCLE $= \ell^2_-(E)$.

If $\underbrace{\theta}_{\text{is an}} : a \to z$ flow $\left[\begin{array}{l} d^*\theta(v) = 0 \quad \forall v \neq a, z \\ d^*\theta(a) = -d^*\theta(z) \end{array}\right]$. We get a current flow $\underbrace{\text{by projecting}}_{\text{with the same strength}}$ it to STAR.

Reciprocity For two distinct edges $e = e'$ (so $\vec{e} \neq \vec{e'} \neq \overleftarrow{e}$),

$\langle P_{STAR}\chi^e, \chi^{e'}\rangle = \langle i^e, \chi^{e'}\rangle = i^e(e')r_{e'}$

i^e = unit current flow from e^- to e^+.

But P_{STAR} is self adj., so it obviously is equal to $\langle \chi^e, P_{STAR}\chi^{e'}\rangle = i^{e'}(e)r_e$.

Denote ~~that now by~~ $y(e, e') = i^e(e')$, then $y(e, e')r_{e'} = y(e', e)\cdot r_e$. Y is called the Transfer current Matrix.

Contraction of $e \in E$ is a new graph G/e obtained by removing e and identifying v_-, v_+. $G\backslash e$ will mark G after deleting e. T_G = UST of G.

Note T_G given $e \in T_G$ is distributed like G/e

" " $e \notin$ " " like $G\backslash e$

example $P(e, f \in T) = P(e \in T)P(f \in T \mid e \in T) = P(e \in T)P(f \in T/e)$

Cor. $P(e, f \in T) \leq P(e \in T)P(f \in T)$, by Kirchhoff's formula & Rayleigh Monotonicity.

What does contraction do to $\ell^2_-(E)$?

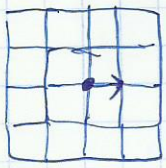
Let i^e be the UCF in G, ~~the~~ $e^- \to e^+$. Let $F \subseteq E$ s.t. $e \notin F$, and won't close ~~a~~ loop ~~...~~ in $(\{e\} \cup F)$. Write $\hat{i}^e$ for UCF $e^- \to e^+$ in G/F.

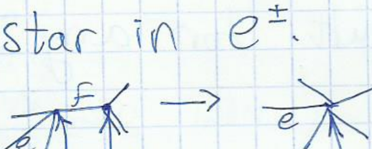
Let $Z = \text{span}\{i^f : f \in F\}$.

Claim: $\hat{i}^e = P_{Z^\perp} i^e_{|E/F}$. We'll check $P_{Z^\perp} i^e_{|F} = 0$. [or, if we keep contracted edges as loops, $\hat{i} = P_{Z^\perp} i^e$.]

Pf: Since $i^e \in \mathrm{STAR}$, $P_Z i^e \in \mathrm{STAR}$, $P_{Z^\perp} i^e \in \mathrm{STAR}$ so the cycle $\overbrace{\text{law}}^{\text{holds}}$. To verify $P_{Z^\perp} i^e_{|F} = 0$, $\langle P_{Z^\perp |F}\, i^e, \chi^f \rangle = \langle i^e, i^f \rangle = 0$.

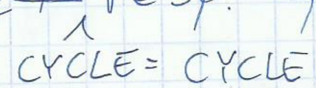
To see the node law of $P_{Z^\perp} i^e$ holds in G/F we write $P_{Z^\perp} i^e = i^e - \sum_{f \in F} a_f i^f$. Note that a star ψ in G/F is a sum θ of stars in G s.t. $\theta(f) = 0$ for $\forall f \in F$. i^e is orthogonal to any star ~~...~~ at $v \notin e^\pm$. So ~~...~~ $i^e_{|E/F} \perp \psi$. Similarly $i^f_{|E/F} \perp \psi \Rightarrow$ where ~~...~~ e, f don't share a vertex, and ψ is not a star at e^- or e^+, $P_{Z^\perp} i^e \perp \psi$. Mass out of e^- is 1 in i^e, 0 in i^f for a star in $e^\pm$. What if e, f share a vertex?





star here + star here = star here and the mass ~~...~~ out of e^- is 1 ~~...~~ in i^e, 0 in i^f.

A different pf. $\hat{i}^e = P_{Z^\perp} i^e$ [G/F has same edges as G, F turned to loops]. Let $\widehat{\mathrm{STAR}}$ and $\widehat{\mathrm{CYCLE}}$ be the resp. spaces in G/F. We'll find $\widehat{\mathrm{CYCLE}} = \mathrm{CYCLE} + \text{span}\{\chi^f \mid f \in F\}$, so $\mathrm{CYCLE} \subseteq \widehat{\mathrm{CYCLE}}$ and hence $\widehat{\mathrm{STAR}} \subseteq \mathrm{STAR}$.



$\widehat{\mathrm{STAR}} \perp \widehat{\mathrm{CYCLE}} \Rightarrow$ $P_{\mathrm{STAR}}\, \widehat{\mathrm{CYCLE}} = \mathrm{STAR} \cap \widehat{\mathrm{CYCLE}}$, so

$P_{\mathrm{STAR}}\, \text{span}\{\chi^f \mid f \in F\} = Z = \text{span}\{i^f \mid f \in F\}$ and $\mathrm{STAR} = \widehat{\mathrm{STAR}} \oplus Z$.

So $\ell^2_-(E) = \widehat{STAR} \oplus CYCLE \oplus Z$. Now, $\hat{i}^e = P_{\widehat{STAR}}\chi^e = P_{Z^+}\underbrace{P_{STAR}\chi^e}_{= i^e}$

<u>The transfer-current thm.</u>

$$\mathbb{P}(e_1,\dots e_k \in T) = \det\{Y(e_i,e_j)\}_{1\le i,j\le k}$$

pf. If $e_1,\dots e_k$ contains a cycle then the LHS is obviously 0. Write $\sum_{j=1}^{k} a_j \chi^{e_j}$ $\overbrace{a_j \in \{-1,0,1\}}^{\text{not all } 0}$ for the cycle. $\forall m=1,\dots k$, $\sum_{j=1}^{k} a_j r_{e_j} \underbrace{i^{e_m}(e_j) = 0}_{\text{cycle law for } i^{e_m}}$, so $\{Y(e_i,e_j)\}_{1\le i,j\le k}\begin{pmatrix} \vdots \\ r_{e_j}a_j \\ \vdots \end{pmatrix} = 0$.

Proceed by induction.

$k=1$ is Kirchhoff's formula. Put $Y_m = \{Y(e_i,e_j)\}_{1\le i,j\le m}$ for $m=1,\dots,k$. $\mathbb{P}(e_1,\dots e_k\in T) = \mathbb{P}(e_1,\dots, e_{k-1}\in T)\,\mathbb{P}(e_k\in T_{G/\{e_1,\dots e_{k-1}\}}) =$
$= \det Y_{k-1}\cdot \hat{i}^{e_k}(e_k) \stackrel{?}{=} \det Y_k$. We know $\hat{i}^{e_k} = P_{Z^+} i^{e_k}$, where $Z = \mathrm{span}\{i^{e_1},\dots, i^{e_{k-1}}\}$. So $\hat{i}^{e_k} = i^{e_k} - \sum_{j=1}^{k-1}\alpha_j i^{e_j}$.

$$Y_k = \begin{pmatrix} — i^{e_1} — \\ \vdots \\ — i^{e_k} — \end{pmatrix} \xrightarrow{i^{e_k}-\alpha_1 i^{e_1}} \begin{pmatrix} — i^{e_1} — \\ \vdots \\ — i^{e_k}-\alpha_1 i^{e_1} — \end{pmatrix} \xrightarrow{i^{e_k}-\alpha_2 i^{e_2}} \dots \xrightarrow{i^{e_k}-\alpha_{k-1} i^{e_{k-1}}} \begin{pmatrix} — i^{e_1} — \\ \vdots \\ — \hat{i}^{e_k} — \end{pmatrix}.$$ But

$\hat{i}^{e_k} = (0,0,0,\dots,\hat{i}^{e_k}(e_k))$ so $\det Y_k =$

$$\det\begin{pmatrix} \overbrace{\begin{matrix} — i^{e_1} — \\ \vdots \\ — i^{e_{k-1}} — \end{matrix}}^{=Y_{k-1}} \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \hat{i}^{e_k}(e_k) \end{pmatrix} = \det Y_{k-1}\cdot \hat{i}^{e_k}(e_k),$$ as we wanted to prove.

We saw $\mathbb{P}(f\in T \mid e\in T) \le \mathbb{P}(f\in T)$, for $e\ne f$. We can expect a similar result for any increasing event.