12/11/14
W3D4

Rand. Spanning Trees

# The Linear Algebra View of Electric Networks

Goal ~~Transfer-~~ ~~cu~~current Thm

$G$ network, $\forall e_1, \dots e_k$ distinct edges, $\mathbb{P}(e_1, \dots e_k \in UST) = \det[\{y(e_i, e_j)\}_{1 \le i,j \le k}]$. $y(e_i, e_j) =$ current ~~throu~~ though $e_j$ in a unit current flow trough $e_i$ (direction won't matter $\vec{e_i}$ instead of $\overleftarrow{e_i}$ will negate 1 collumn and 1 row so the determinant will stay the same)

$\ell^2_\pi(V)$ - functions on vertices with $\langle f, g \rangle_\pi = \sum_{x \in V} \pi_x f(x) g(x)$.

$\ell^2_-(\vec{E})$ - antisymm. func. on $\vec{E}$. $\langle \theta, \theta' \rangle = \frac{1}{2} \sum_{e \in \vec{E}} r_e \theta(e) \theta'(e) = \sum_{e \in E_{1/2}} r_e \theta(e)\theta'(e)$ ← each edge appears in 1 arbitrary direction.

(Both spaces are of real-valued functions.)

The gradient operator $d: \ell^2(V) \to \ell^2_-(\vec{E})$ is defined by $(df)(e) = c_e(f(e_+) - f(e_-))$ (dest. & source of $e$)

The divergence operator $d^*: \ell^2_-(E) \to \ell^2(V)$ is defined by $(d^*\theta)(v) = \frac{1}{\pi_v} \sum_{\substack{e \in E \\ e^- = v}} \theta(e)$

$\langle \theta, df \rangle_r = \frac{1}{2} \sum_{e \in E} \cancel{r_e} \theta_e \cancel{c_e} (f(e_+) - f(e_-))$

$\langle d^*\theta, f \rangle_\pi = \sum_{x \in V} \cancel{\pi_x} f(x) \frac{1}{\cancel{\pi_x}} \sum_{y \sim x} \theta(\vec{xy})$

So we see $\langle \theta, df \rangle_r = - \langle d^*\theta, f \rangle_\pi$.

$f: V \to \mathbb{R}$ is harmonic at $v$ iff $(d^*(df))(v) = 0$

Networks If $i$ is a current flow $a \to z$ correspondind to voltage $v$: $(d^* i)(x) = 0$ $\forall x \ne a, z$, $\text{strength}(i) = (d^* i)a$, $d(v)(e) = i(e)$.

Given a directed edge $e$, $\chi^e = \mathbb{1}_{\vec{e}} - \mathbb{1}_{\overleftarrow{e}}$.

A star at $x$ is the element of $\ell^2_-(E)$ $\sum_{e: e_+ = x} c(e) \chi^e$.

STAR = span {all stars in $v \in V$} subspace of $\ell^2_-(E)$.

Note that STAR = $d\,\ell^2(V)$, since $\sum_{e: e_+ = x} c(e) \chi^e = d\,\mathbb{1}_{(x)}$.

If $e_1, \dots e_k$ is an oriented cycle the $\ell^2_-(E)$ elem. $\sum_{i=1}^{k} \chi^{e_i}$.

CYCLE = span {all cycles}. subspace of $\ell^2_-(E)$.

<u>Note</u> STAR $\perp$ CYCLE. In fact STAR $\oplus$ CYCLE $= \ell^2_-(E)$ -

cause if $\theta \perp$ CYCLE $\rightarrow$ define $F: V \rightarrow \mathbb{R}$ by

$F(v) = \sum_{j=1}^{m} r_{e_i} \theta(e_i)$ - $e_1, \dots e_m$ is a directed path $a \rightarrow v$.

Then $\theta = dF \rightarrow \theta \in$ STAR.

Alternatively $\theta \in$ (STAR $\cap$ CYCLE) must be harmonic everywhere.

If $\theta \perp$ STAR $\leftrightarrow$ $(d^*\theta)(v) = 0$ so ~~STAR~~ CYCLE $= \ker d^*$.

If $i$ is a current flow $a \rightarrow z$ and $\theta$ is a flow $a \rightarrow z$ with same strength

$d^*\theta \equiv d^* i$, then $d^*(\theta - i) = 0$ so $\theta - i$ is a cycle. $\theta = i + (\theta - i)$

[this is the orthogonal decomposition]. $\underbrace{\|\theta\|_r^2}_{\mathcal{E}(\theta)} = \underbrace{\|i\|_r^2}_{\mathcal{E}(i)} + \|\theta - i\|_r^2$

and we get Thomson's law, restated. - $\mathcal{E}(\theta) > \mathcal{E}(i)$

unless $\theta \equiv i$.