5/11/14
V2D4

Random Spanning Trees

Thompson's principle $R_{eff}(a\leftrightarrow z) = \inf\{\mathcal{E}(\theta) \mid \theta\ \ a\to z \text{ flow}, \|\theta\|=1\}$. Where,

$$\mathcal{E}(\theta) = \sum_{e\in E} r_e \theta^2(e)$$

Cor. When the graph increases (or r_e decreases), R_{eff} decreases. Formally:

Rayleigh's monotonicity If $\{r_e\}$ and $\{r'_e\}$ are resistances on the same graph with $\forall e\in E\ r_e \le r'_e$, $R_{eff}(a\leftrightarrow z;\{r_e\}) \le R_{eff}(a\leftrightarrow z;\{r'_e\})$

Cor. If $r_e \equiv 1$, $R_{eff}(a\leftrightarrow z) \le$ (length of shortest path from a to z),

Tight for 

Infinite ~~Resistance~~ Networks

Let G be an infinite conn. graph and $\rho\in G$.

Let $\{G_n\}_{n\in\mathbb{N}}$ be a finite exhaustion $\rho\in G_1\subseteq G_2 \dots \bigcup_{n\in\mathbb{N}} G_n$, $\forall n\in\mathbb{N}\ |G_n|<\infty$.

Form finite graphs for each n by replacing $G\setminus G_n$ by a single vertex z_n (maintain the edges from G_n to $G\setminus G_n$)

$R_{eff}(\rho\leftrightarrow\infty) = \lim_{n\to\infty} R_{eff}(\rho\leftrightarrow z_n)$. Exists by monotonicity.

We claim it won't depend on G_n. If we have A_n, B_n - they intertwine (for large enough m, $A_n\subseteq B_m$ ...).

So $\mathbb{P}$(ρ-starting random walk never returns to ρ) $= \dfrac{1}{\pi_a R_{eff}(\rho\leftrightarrow\infty)}$, therefore G is recurrent $\leftrightarrow R_{eff}(\rho\leftrightarrow\infty)=\infty$.

Prop. The following are equivalent:

1) Weighted RW is transient.
2) $\exists a\in V$ $R_{eff}(a\leftrightarrow\infty)<\infty$.
3) $\exists$ flow θ from a to ∞ (antisymmetric and node law for $V\setminus\{a\}$)

The Nash-Williams criterion/inequality

If $\{\Pi_n\}$ are disjoint edge cutsets saperating a,z in a finite graph G (every path $a\to z$ intersects Π_n), $R_{eff}(a\leftrightarrow z) \ge \sum_n \left(\sum_{e\in\Pi_n} c_e\right)^{-1}$.

If it's $=\infty$ in infinite G $\Updownarrow$ G is ~~transient~~ recurrent

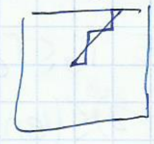
Proof Let θ be a unit flow $a\to z$. $\sum_{e\in\Pi_n} c_e \cdot \sum_{e\in\Pi_n} r_e\theta^2(e) \overset{C\text{-}S}{\ge} \left(\sum_{e\in\Pi_n} \theta(e)\right)^2 \ge \|\theta\|^2 = 1$,

so, $\mathcal{E}(\theta) \ge \sum_n \sum_{e\in\Pi_n} r_e \theta^2(e) \ge \sum_n \left(\sum_{e\in\Pi_n} c_e\right)^{-1}$ and the result follows.

Example $\mathbb{Z}^2$ is recurrent. $G_n = n \times n$ box. Π_n = edges between G_n, $G_{n+1} \setminus G_n$. $|\Pi_n| = \Theta(n)$. So $R_{eff}(\rho \leftrightarrow Z_n) \geq \sum_{i=1}^{n} \frac{c_1}{i} = c_2 \log n$.

Random Path Method

Let μ be a prob. measure on paths from a to z. Then $\mathbb{E}_\mu[\#\text{traversed } \vec{e} - \#\text{travesed } \overleftarrow{e}] = \theta(\vec{e})$ is a flow with $\|\theta\| = 1$. If we take



where the line chosen uniformally. $\mathbb{P}(\text{chose } \vec{e}) = \theta\left(\frac{1}{k}\right)$

$\vec{e}$: $G_{k-1} \to G_k \setminus G_{k-1}$