4/11/14

W2D3

~~Geometry of Numbers~~

Rand. Spanning Trees

Review: G finite, conn. graph. Edge weights c_e conductances, $r_e = \frac{1}{c_e}$ resistances, $h: V \to \mathbb{R}$ harmonic (at x) if $h(x) = \sum_y \frac{c_{xy} h(y)}{\pi_x}$ where $\pi_x = \sum_y c_{xy}$. A flow (from a to z) $\theta : \vec{E} \to \mathbb{R}$ s.t. $\theta(\vec{xy}) = -\theta(\vec{yx})$, and $\underbrace{\sum_y \theta(\vec{xy}) = 0}_{\text{for all } x \neq a,z}$. Cycle law is satisfied by a flow θ if $\vec{e}_1, \dots, \vec{e}_n$ is a closed cycle, $\sum r_{e_i} \theta(\vec{e}_i) = 0$. V is a voltage if it's harmonic on $V / \{a, z\}$. Given a voltage V we define the current flow $I(\vec{xy}) = c_{xy}(V(y) - V(x))$ – it satisfies the cycle law.

Assume $V(a) \leq V(z)$. The strength of a flow θ is $\|\theta\| = \sum_x \theta(\vec{ax})$.

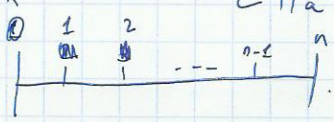
Prop: if θ is a flow from a to z (that satisfies the cycle law), I is a current flow and $\|\theta\| = \|I\|$ then $\theta \equiv I$.

Proof: Put $J = \theta - I : \vec{E} \to \mathbb{R}$ a flow satisfying the node law everywhere (even at a, z) at the cycle law. So, define the harmonic function $h(x) = \sum_{i=1}^{n} J(\vec{e}_i) r_i$ where $e_1, \dots, e_n$ a path from a to x. h is harmonic everywhere with $h(a) = 0$, so $h \equiv 0$, and therefore $\theta \equiv I$.

So, the number $\frac{V(z) - V(a)}{\|I_V\|}$ ← induced by V doesn't depend on the choice of V. We call it the effective resistance between a & z is G, $R_{\text{eff}}(a \leftrightarrow z)$.

Prob. inter. $P_a\left(\begin{smallmatrix}\text{hit } z \text{ before} \\ \text{returning to } a\end{smallmatrix}\right) = \sum \frac{c_{ax}}{\pi_a} P_x\left(\begin{smallmatrix}\text{hit } z \\ \text{before } a\end{smallmatrix}\right) = \sum \frac{c_{ax}}{\pi_a} \frac{V_x - V_a}{V_z - V_a} =$

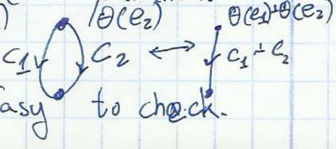
$$= \frac{1}{\pi_a} \frac{\|I\|}{V_z - V_a} = \frac{1}{\pi_a R_{\text{eff}}(a \leftrightarrow z)}.$$



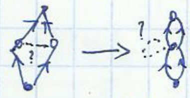
$P\left(\begin{smallmatrix}\text{return to } 0 \text{ before} \\ \text{reaching } n\end{smallmatrix}\right) = 1 - \frac{1}{n} = \frac{n-1}{n}.$

Network Simplifications: 1) Parallel law – conductances add in parallel.

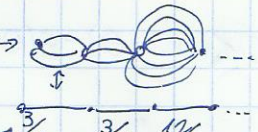
If e_1, e_2 are parallel edges, have equal $R_{\text{eff}}(a \leftrightarrow z)$. Easy to check.



2) Series law - resistances add in series $\to \xrightarrow{r_1}\xrightarrow{r_2} \to \xrightarrow{r_1+r_2}$

3) Gluing - identifying v's with equal voltage. 

<u>Example</u> Let Γ be a spherically symm. tree. That is - all v's with same height have same degree and resistances.

For example,  Γ_n - v's in level n. 

$\xrightarrow{3/r_1}\xrightarrow{3/r_2}\xrightarrow{12/r_3}$ ...

$\frac{r_1}{3}+\frac{r_2}{3}+\frac{r_3}{12}+\dots+\frac{r_n}{|\Gamma_n|}+\dots$

$R_{eff}(\rho \leftrightarrow \Gamma_n) = \sum_{i=1}^{n} \frac{r_i}{|\Gamma_i|}$. This Γ is <u>recurrent</u> iff $\sum_{i=1}^{\infty} \frac{r_i}{|\Gamma_i|} = \infty$.

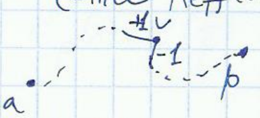
<u>Def.</u> The <u>unit</u> current flow is the current flow with $\|\theta\| = 1$.

Let G be a simple finite conn. graph with unit resistances, and a, z marked v's and choose a UST T of G, and for $\vec{e}$ define

$f(\vec{e}) = \mathbb{E}\left[\begin{matrix}\#\text{ net crossings } (\pm 1 \text{ or } 0) \\ \text{of } e \text{ in the unique path from } a \text{ to } z\end{matrix}\right]$. +1 if $\vec{e}$ is used and -1 if $\overleftarrow{e}$ is.

<u>Thm.</u> $f(e)$ is the unit current flow.

<u>Cor.</u> $P((x,y) \in T\ UST) = R_{eff}(x \leftrightarrow y)$. (since $R_{eff}(x \leftrightarrow y) = V_y - V_x = I_{\vec{xy}}$).

<u>Pf.</u> f is antysym. Node law:  So in expectation...

Unit flow $a \xrightarrow{+1} \cdots \xrightarrow{-1} z$. Just need to verify the cycle law - $\vec{e}_1 \dots \vec{e}_n$ a cycle $\to \sum_{i=1}^{n} f(\vec{e}_i)$. Fix $i \in \{1,\dots,n\}$ and consider a map from $\{T$ spanning tree which uses $\vec{e}_i$ in the $\vec{az}$ path$\}$ to $\{t'$ uses $\overleftarrow{e}_j$ for some $j \neq i$ and doesn't use $\vec{e}_i$ in the $\vec{az}$ path$\}$ which is a bijection on it's image ($\leftrightarrow$ surjective).

- Erase $\vec{e}_i$ from T. That splits it to 2 conn. comp. - take the first $\overleftarrow{e}_j$ in the cycle after $\vec{e}_i$ that connects them.

Given e_j, e_i it's reversible. So $\sum_{i=1}^{n} f(e_i) = \sum_T \frac{1}{\#_{sp}\text{trees}} \sum_{i=1}^{n} g_i(T) = 0$.

<u>Thompson's law</u> - The current flow in the one with minimal energy.

$R_{eff}(a \leftrightarrow z) = \inf\{\mathcal{E}(\theta) : \theta \text{ is a flow with } \|\theta\| = 1\}$ where $\mathcal{E}(\theta) = \sum_{e \in E} r_e \theta^2(e)$.

There exists a unique minimizer - the unit current flow.

Proof. By compactness exists a minimizer θ which is a flow $\|\theta\|=1$. Let's prove the cycle law ($\to$ unique):

Chose a cycle $\vec{e_1}, \dots \vec{e_n}$. Let γ be the flow with $\gamma(\vec{e_i})=1$ and $\gamma(\overleftarrow{e_i})=-1$, with 0 everywhere else $\forall \varepsilon>0$,

$$0 \leq \mathcal{E}(\varepsilon\gamma+\theta)-\mathcal{E}(\theta)=\sum_{i=1}^{n} r_{e_i}\left[(\theta(\vec{e_i})+\varepsilon)^2-\theta^2(\vec{e_i})\right]=\varepsilon^2\sum_{i=1}^{n} r_{e_i}+2\varepsilon\sum_{i=1}^{n} r_{e_i}\theta(\vec{e_i})$$

$$0 \leq \varepsilon\sum_{i=1}^{n} r_{e_i}+2\sum_{i=1}^{n} r_i\theta(e_i) \quad \text{Take } \varepsilon \searrow 0 \quad \text{so } 0 \leq \sum_{i=1}^{n} r_i\theta(e_i).$$

Similarly it's ≤ 0 – so it's 0.

Therefore θ is the unit current flow I.

$$\mathcal{E}(I)=\sum_e r_e I^2(e)=\frac{1}{2}\sum_x\sum_y c_{xy}(V_y-V_x)^2=\frac{1}{2}\sum_{x,y}(V_y-V_x)I(\vec{xy})=$$

$$\frac{1}{2}\sum_{x,y} V_y I(\vec{xy})-\frac{1}{2}\sum_{x,y} V_x I(\vec{xy})=\sum_{x,y} V_x I(\vec{xy})=\sum_{y\neq a,z}\underbrace{\sum_x V_x I(\vec{xy})}_{=0}=V(a)-V(z)=R_{eff}$$

between a and z, since $\|I\|=1$.