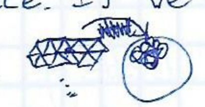
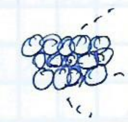
V13D4
21/1/15

# Random Spanning Trees

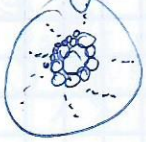
A circle packing P is a collection of circles in the plane with disjoint interiors. G(P) tangency graph - V = circles and E = pairs of tangent circles, so the E(G) is planar.

Thm. (Koebe '36) $\forall$ G finite planar has a corresponding circle packing P with $G(P) \cong P$.
This P is unique up to Mobius transformation (this is equivalent to the Riemann Mapping Thm.).

Idea of equivalence - triangulation of the set except for the outerface $\Rightarrow$ CP with outer circles tangent to the outer face. If we use arbitrarily small "pyramidal" triangulation  and using smaller & smaller lattices we'll get a conformal mapping. How do we map?

Thm (Sullivan) P a CP s.t. G(P) is the triangular lattice than its  up to Mobius (rigidity of circle packing). A general result for triangulation was proven by Schramm.

only scaling & trans. if we disallow accumulance in a single pt.

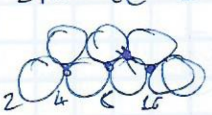
Negative example: He-Schramm '95 

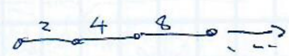
G one-ended triangulation w. bdd. degree, define $Carr(P) = \{$union of circles + spaces between 3 circles$\}$, then

1) If $Carr(P) = \mathbb{R}^2$ G is recurrent, if $Carr(G) \subsetneq \mathbb{R}^2 \Rightarrow$ transient

2) If G transient, $\forall$ s.c. domain $D \neq \mathbb{R}^2$ $\exists P$ w. $Carr(P) = D$.

In the $1^{st}$ part of 1, bdd. deg is necessary (not for $2^{nd}$!)



looks like $\mathbb{Z}^2$ with a single line 

idea for $1^{st}$ part of 1:

$$\frac{1}{R_{eff}} = \inf\{\mathcal{E}(f) \mid f(a)=0,\ f(z)=1\}.$$

so resistance $\longrightarrow \infty$.

Bound using $f(x) = \frac{dist(x, \partial B)}{2r}$

$(f(y)-f(x))^2 \leq \frac{(d(x,y))^2}{r^2} \leq \frac{Cr_x^2}{r^2}$ and we get resistance indep. of r.