W13D3
20/1/15

Random Spanning Trees

Spectral Radius and Non-Amenability

G d-regular & connected. $\{X_n\}$ SRW on G. $P=\{p(x,y)=\frac{1}{d}\mathbb{1}_{(x\sim y)}\}$ is an operator $P:\ell^2(V)\to\ell^2(V)$ by $(Pf)(x)=\mathbb{E}_x f(X_1)$,

$\|P\|=\sup_{f\in\ell^2(V)}\left(\frac{\|Pf\|_2}{\|f\|_2}\mid f\neq 0\right)$ the operator norm. equal in fin. graphs, $y\equiv 1$ is in $\ell^2(V)$.

<u>Observation 1</u> $\|P\|\le 1$ (as the sum on each row in 1).

<u>pf</u> $\|Pf\|_2^2=\sum_x\left(\sum_{y\sim x}P(x,y)f(y)\right)^2\overset{cs}{\le}\sum_x\underbrace{\left(\sum_{y\sim x}P(x,y)\right)}_{1}\left(\sum_{y\sim x}P(x,y)f^2(y)\right)=\sum_x\sum_{y\sim x}\frac{f(y)^2}{d}=\sum_y f(y)^2=\|f\|_2^2.$

<u>Note</u> $(P^nf)(x)=\mathbb{E}_x f(X_n)$
(is just $\sum_y P_n(x,y)f(y)$).

<u>observation 2</u> P is self-adjoint $\langle Pf,g\rangle=\sum_x g(x)\sum_{y\sim x}P(x,y)f(y)=$
$1/d\sum_{x\sim y}f(y)g(x)$ is symmetric in f,g.

<u>Prop</u> $\|P\|=\sup_{f\in\ell^2(V)}\left\{\frac{|\langle Pf,f\rangle|}{\langle f,f\rangle}\mid f\neq 0\right\}$. Rayleigh Quotient

<u>pf.</u> $\langle Pf,f\rangle\le\|Pf\|_2\cdot\|f\|_2\le\|P\|\cdot\|f\|_2^2$ so if the sup above is denoted by c, we get $c\le\|P\|$. Conversely $\forall f$, $\langle Pf,f\rangle\le c\langle f,f\rangle$
and $|\langle Pf,g\rangle|=\frac{|\langle P(f+g),f+g\rangle-\langle P(f-g),f-g\rangle|}{4}\le\frac{c}{4}\left[\langle f+g,f+g\rangle+\langle f-g,f-g\rangle\right]=$
$\frac{c}{2}\left(\|f\|_2^2+\|g\|_2^2\right)$. So, if we choose $g=(Pf)\cdot\frac{\|f\|_2}{\|Pf\|_2}$ so $\langle g,g\rangle=\|f\|_2^2$,
and $|\langle Pf,g\rangle|=\frac{\|f\|_2}{\|Pf\|_2}\|Pf\|_2^2$ so we get $\|f\|_2\|Pf\|_2\le c\|f\|^2$ and therefore $\|P\|\le c$ as well.

<u>Prop</u> $\|P\|=\limsup_{n\to\infty}(P_n(x,y))^{1/n}$. [Convince yourself: this is indep. of x,y].
and $P_n(x,y)\le\|P\|^n$ $\forall x,y,n$. Spectral Radius.

<u>pf.</u> $p_n(x,y)=\langle\mathbb{1}_{(x)},P^n\mathbb{1}_y\rangle\overset{c\text{-}s}{\le}\|P^n\mathbb{1}_{(y)}\|_2\le\|P^n\|\le\|P\|^n$. So, if we denote the limsup above by ρ, we got $\rho\le\|P\|$. For the converse,
$\|P^{n+1}f\|_2^4=\left(\langle P^nf,P^{n+2}f\rangle\right)^2\overset{c\text{-}s}{\le}\|P^nf\|_2^2\,\|P^{n+2}f\|_2^2$ so we get
$\frac{\|P^{n+2}f\|^2}{\|P^{n+1}f\|^2}\ge\frac{\|P^{n+1}f\|^2}{\|P^nf\|^2}\nearrow L^2$, and $\left(\prod_{n=0}^{N-1}\frac{\|P^{n+1}f\|}{\|P^nf\|}\right)^{1/N}=\left(\frac{\|P^Nf\|}{\|f\|}\right)^{1/N}\xrightarrow{N\to\infty}L$

(eq. $\|P^nf\|^{1/n}\to L$). Since $p_{2n}(x,x)\ge p_n^2(x,x)$ we get
$\rho=\limsup_{n\to\infty}p_{2n}(x,x)^{1/2n}$. Now, $L=\limsup_{n\to\infty}\|P^nf\|^{1/n}=\limsup_{n\to\infty}\langle P^{2n}f,f\rangle^{1/2n}=$
$\limsup\left(\sum f(x)f(y)p_{2n}(x,y)\right)^{1/2n}$. For $f\ge 0$ fin. supported this is

Def $\Phi_E = \inf_{\substack{K \subseteq V \\ \text{finite}}} \frac{|\partial_E K|}{d|K|}$ (normalized so that $0 \leq \Phi_E \leq 1$).

Thm (Cheeger) $\Phi_E^2(G)/2 \leq 1-\rho(G) \leq \Phi_E(G)$.

Note that $1-\rho(G) = \inf_{f \in \ell^2(V)} \left\{ \frac{\langle (I-P)f, f\rangle}{\langle f, f\rangle} \mid f \neq 0 \right\}$.

pf. Let $f = \mathbb{1}_K$. Then $1-\rho(G) \leq \frac{\langle (I-P)f, f\rangle}{\langle f, f\rangle} = \frac{1}{|K|} \sum_x f(x)\left(f(x) - \sum_{y \sim x} \frac{f(y)}{d}\right) =$

$\frac{1}{|K|}\left(|K| - \frac{1}{d} \sum_x \sum_{y\sim x} f(x) f(y)\right) = \frac{|K| - \frac{1}{d}[d|K| - |\partial_E K|]}{|K|} = \frac{|\partial_E K|}{d|K|}$. Taking inf over $K \subseteq V$ finite we get $1-\rho \leq \Phi_E$.

Lemma (Poincare) If $f \geq 0$ finitely supported, $\Phi_E d \sum_x f(x) \leq \sum_{e=(x,y)\in E} |f(x) - f(y)|$.

pf. For $t > 0$ take $K = \{x \mid f(x) > t\}$. By definition $\Phi_E d|K| \leq |\partial_E K| =$
$\sum_{e=(x,y)\in E} \mathbb{1}_{f(x) > t \geq f(y)}$. Integrating over t from 1 to infinity we get the thm.

Now, we want to prove $1-\rho \geq \underbrace{1-\sqrt{1-\Phi_E^2} \geq \frac{\Phi_E^2}{2}}$. Let f be ~~compact~~ finitely supported. By lemma $\left(\sqrt{1-x} \leq 1 - \frac{x}{2}\right)$ on f^2 we get

$d^2\langle f, f\rangle^2 \leq \Phi_E^{-2}\left(\sum_{e=(x,y)\in E} |f^2(x) - f^2(y)|\right)^2 = \Phi_E^{-2}\left(\sum_{e=(x,y)\in E} |f(x)+f(y)|\cdot|f(x)-f(y)|\right)^2 \overset{C-S}{\leq}$
$\Phi_E^{-2} \sum_{e=(x,y)\in E} (f(x)-f(y))^2 \sum_{e=(x,y)\in E} (f(x)+f(y))^2$. Now, $d\langle f, (I-P)f\rangle = d\sum_x f(x)\left[f(x) - \sum_{y\sim x} \frac{f(y)}{d}\right] =$

$\sum_x \sum_{y\sim x} f^2(x) - f(x)f(y) = \sum_{e=(x,y)\in E} (f^2(x) + f^2(y) - 2f(x)f(y))$ and $\sum_{e=(x,y)\in E} (f(x)^2 + f(y)^2 + 2f(x)f(y)) =$

$d\sum_x f^2(x) + d\sum_x f(x) \sum_{y\sim x} \frac{f(y)}{d} = d[\langle f, f\rangle + \langle f, Pf\rangle] = d[2\langle f, f\rangle - \langle (I-P)f, f\rangle]$ so for

any such f we get $\langle f, f\rangle^2 \leq \Phi_E^{-2} \langle f, (I-P)f\rangle [2\langle f, f\rangle - \langle (I-P)f, f\rangle]$ so

$\Phi_E^2 \leq \underbrace{\frac{\langle f, (I-P)f\rangle}{\langle f, f\rangle}}_{R} \underbrace{\left(2 - \frac{\langle f, (I-P)f\rangle}{\langle f, f\rangle}\right)}_{2-R}$ so $1 - \Phi_E^2 \geq 1 - 2R + R^2 = (1-R)^2$ and

we get $R \geq 1 - \sqrt{1-\Phi_E^2}$ for arbitrary f, and thus $\rho(G) \geq 1 - \sqrt{1-\Phi_E^2}$.