V12D4
14/1/15

Random Spanning Trees

$U: E \to \mathbb{R}_+$ injective. $WMSF = \bigcup_{x \in V} I(x)$. $\forall x,y$ either $I(x) \cap I(y) = \phi$ or $|I(x) \Delta I(y)| < \infty$ (either they're in the same component of WMSF or not). We used those facts to show $\theta(p_c) = 0 \Rightarrow$ WMSF is one ended. We can show that this is the case for edge-nonamenable Cayley graphs: $\Phi_E = \inf_{\substack{K \subset V \\ \text{finite}}} \frac{|\partial_E K|}{|K|} > 0$. Put $\forall K \subseteq V$ fin, $\alpha_K = \frac{1}{|K|} \sum_{x \in K} \deg_K(x)$, $\alpha = \sup \alpha_K$.

Then $\alpha + \Phi_E =$ avg. degree since $\sum_{x \in K} (\deg_K(x) + |\partial_K(x)|) = d \cdot |K|$, so $\alpha + \Phi_E \leq d$ and also $\alpha_K + \frac{|\partial_E K|}{|K|} \geq d \Rightarrow \alpha + \Phi_E \geq d$.

Thm G cayley, P inv. prob measure on 2^E. If a.s. all clusters are finite, then $\mathbb{E} \deg_\omega(0) \leq d - \Phi_E$.

pf. Send mass $\frac{\deg_\omega x}{|C(x)|}$ to all $y \in C(x)$. Mass out - $\deg_\omega 0$.
Mass in $\frac{1}{|C(x)|} \sum_{y \in C(x)} \deg_\omega y \leq \alpha = d - \Phi_E$.

Thm G Cayley & nonamenable, then $\theta(p_c) = 0$.

pf. We proved #inf. clusters is either 0, 1, ∞ a.s.
Assume 1 - $\exists!$ inf. cluster in $\{e \mid U(e) \leq p_c\} = W_{p_c}$, say W'. $\forall x \in V$ let $W(x) \in W'$ be the closest to x (if there are > 1 pick u.a.r.). Let $\epsilon > 0$, $\eta_\epsilon = \{(x,y) \in E \mid W(x), W(y)$ belong in same component of $W_{p_c - \epsilon}\}$. Components of $W_{p_c} \to \eta_\epsilon$ are finite so $\mathbb{E} \deg_{\eta_\epsilon}(0) \leq d - \Phi_E$ but $\bigcup_\epsilon \eta_\epsilon = E(G)$ and $\eta_\epsilon \subseteq \eta_{\epsilon'}$ for $\epsilon' < \epsilon$, so by the dominated convergence theorem $\Rightarrow \Phi_E = 0$.

prop 1 $p_c(W_{p_c}) = 1$ (just double-exposure $p_c, 1-\epsilon \Leftrightarrow p_c(1-\epsilon)$ single exposure)

prop 2 If $\mathcal{F}$ invariant forest, $\mathbb{E}(\deg_{\mathcal{F}}(0) \mid 0 \in \mathcal{F}) > 2$ then $p_c(\mathcal{F}) < 1$ a.s.
since for $\mathcal{F}$ p-perc p close to 1 also holds $\Rightarrow \exists$ infinite cluster in $\mathcal{F}'$.

Assume $\infty \Rightarrow \exists$ furcation pts. with prob. > 0. Consider $\mathcal{F} =$ FMSF on W_{p_c} only on infinite clusters.
with probability > 0 0 is a furcation in $\mathcal{F}$, so $\mathbb{E}[\deg_{\mathcal{F}}(0) \mid 0 \in \mathcal{F}] > 2$.
Both propositions now yield a contradiction.