13/1/15
W12D3

~~The~~ Random Spanning Trees

G Cayley $f: G\times G \to \mathbb{R}$ invariant $[f(x,y)=f(\gamma x,\gamma y)]$ then $\sum_{x\in G} f(0,x) = \sum_{x\in G} f(x,0)$. Consider a prob. measure P on $2^{E(G)\cup V(G)}$ that is invariant $[P(A)=P(\gamma A)]$.

Ex 1. Can P on $2^{V(G)}$ be concentrated one 1 vxs? If $X\subseteq V$ is the random subset, can $P(|X|=1)=1$? No, since $P(X=v)>0$ then it's ~~the~~ equal for any $v\in G$. If $P(|X|=1)$ $P(x\ni v$ ~~&~~ $x\ni u)=0$ so $\sum_v P(X=v)=\infty>1$.

Ex 2. Can $P(|X|=k)=1$? No, create P' by choosing X by P, then choosing $v\in X$ u.a.r. - P' contradicts ex. 1.

Ex. 3 P inv. on $2^{E(G)}$. ~~X is a furcation if removing it~~ # fin. clusters is 0 or ∞, since o/w we can choose $v\in V$ uniformly at random from the union of those clusters, contradicting ex. 1.

Ex 4. Recall $x\in G$ is a furcation of $W\in 2^E$ if removing it splits the component of x to ≥ 3 ∞ components. # of such points in 0 or ∞ with probability 1.

Ex.5 Each cluster has ∞ or 0 furcation points: Let $N(x,W)$ = # of furcations in $C(x)$. Let $F(x,y,W)=\begin{cases} 0 & N(x,W)\in\{0,\infty\} \\ \frac{1}{N(x,W)} & 0<N(x,W)<\infty,\ y \text{ furcation} \\ 0 & \text{o/w}\end{cases}$ $f(x,y)=\mathbb{E}_P[F(x,y,W)]$ is a mass transport. Then $\sum_{y\in G} f(0,y)\leq 1$ but $\sum_{y\in G} f(y,0)\in\{0,\infty\}$, so P(0 furcation in a cluster with $0<$#furcation$<\infty)=0$ - this is true for any vertex so P(the above happens anywhere in G)$=0$.

Prop P inv. on 2^E s.t. W is a.s. a spanning forest with all trees $<\infty$. Then $\mathbb{E}[\deg_W 0]<2$.

pf. $x\to y$ mass $\frac{\deg_W(x)}{|C(x)|}$. Mass out $-\ \mathbb{E}\deg_W 0$. Mass in ~~~~ is $\mathbb{E}$[avg deg. in x's cluster] $<\infty$.

1) If a.s. every tree has 1/2 ends then $\mathbb{E}[\deg_W o \mid o\in F]=2$.

2) If a.s. some tree has 3 ends $\Rightarrow \mathbb{E}[\deg_W o \mid o\in F] > 2$.
$\geq\infty$ ends

pf. Let $\xi(x,y,F)$ be the indicator that $\exists\, \infty$ ray $x \xrightarrow{\text{single edge}} y \to \infty$ in F

$$F(x,y,F)=\begin{cases} 2\xi(x,y,F) & \text{if } \xi(y,x,F)=0 \\ \xi(x,y,F) & \text{o/w} \\ [0 & x\notin F] \end{cases}, \quad f(x,y)=\mathbb{E} \text{ of that.}$$

For $x\in F$, $\sum_y F(o,y,F)+F(y,o,F) = 2\deg_F(0).\ [(\to + \leftarrow)=2]$.

By mass transport, $\mathbb{E}[\deg_F o \mid o\in F] = \sum_x f(o,x)$. If all comps. are 1 or 2 ended the mass out is 2, o/w >2
$2\cdot\mathbb{P}(x\in F)$ $\quad >2\mathbb{P}(x\in F)$

<u>Back to WMSF</u> -

<u>Prop</u> G cayley, $\mathbb{E}\deg_{WMSF}=2$ and therefore each tree has at most 2 ends.

<u>pf.</u> Recall $e\in WMSF \iff U(e) < Z_w(e) = \inf_{\substack{\text{ex. path} \\ P: e^+ \to e^-}} \sup\{U(e') \mid e'\in P\}$.

For $p<1$, let $\eta_p = \{e \mid U(e) \leq p Z_w(e)\}$. η_p is a collection of trees with ≤ 1 end. So $\mathbb{E}\deg_{\eta_p} o \leq 2$. But $WMSF = \bigcup_p \eta_p$ and by dominated convergence $\mathbb{E}\deg_{WMSF} o \leq 2$. By prev. thm, $\forall$ comp. are 1 or 2 ended $\Rightarrow \mathbb{E}\deg_{WMSF}=2$.

<u>Thm</u> G cayley graph with $\theta(p_c)=0$. Than a.s each comp. of WMSF is 1-ended.
$\mathbb{P}_{p_c}(\infty \text{ cluster})=0$

<u>pf.</u> Let $e_1, e_2, \ldots$ be the invasion tree of x. Since $\theta(p_c)=0$, $\forall k \ \sup_{n\geq k} U(e_n) \geq p_c$. Also, recall we proved - a.s. $\forall p > p_c$ $\forall x\in V$, $I(x)$ intersects some ∞ p-cluster, so $\limsup_{n\to\infty} U(e_n) = p_c$. So, $\exists\, \infty$ many k values s.t. $U(e_k) = \sup_{n\geq k} U(e_n)$. For each such k, $\forall n > k$ $U(e_n) < U(e_k)$. So U_k separates x from $\{e_n\}_{n\geq k}$, so $I(x)$ is 1-ended a.s. For any x, y, either $I(x)\cap I(y)=\emptyset$ or $|I(x) \triangle I(y)| < \infty$. Now, $F = \bigcup_x I(x)$, so for each cluster in F $\exists$ uniquely chosen ray. If with $\mathbb{P}>0$, $o\in$ component with two ends, $\{x_n\}_{-\infty}^{\infty}$ the bi-inf. path. $\limsup_{n\to\infty} U(x_n, x_{n+1}) = p_c$, so $\varepsilon = \sup_{\substack{\text{canonical line} \\ \hat{I}}} U(x_n, x_{n+1}) - p_c > 0$.