W11D4
7/1/15

Random Spanning Trees

$\xi_p(x) = $ # edges (y,z) with $y \in I(x)$, $z \in$ an ∞ p-cluster.

Lemma 2 G transitive, $p > p_c$, then $\xi_p(x) = \infty$.

Thm. $\forall p > p_c$, $I(x)$ intersects an ∞ p-cluster.

pf. Take $p > p_1 > p_c$. Given $\{U(e) \mid e \in E\}$, color an edge blue if it lies in an ∞ p_1-cluster, red if it is adjacent to some blue edge (but not blue itself).

Conditioned on all colors, $U(\text{Blue edge}) \sim \text{Uniform}[0, p_1]$, $U(\text{red edge}) \sim \text{Uniform}[p_1, 1]$.

Now consider $I(x)$ conditioned on all colors. we claim: some edge in $I(x)$ is adjacent to some red/~~blue~~ edge with $U(e) \leq p$. This is becaus by lemma 2, $\xi_p(x) = \infty$, so $I(x)$ adjacent to ∞ many colo

When $I(x)$ first becomes adjacent to a colored edge e - the color of e is blue $\to U \sim U[0, p_1]$.
red $\to U \sim U[p_1, p]$

~~The first~~ Prob. at each step is $\frac{p - p_1}{1 - p_1} > 0$ so at least 1 red edge will be in the p-cluster.

The Mass-Transport Principle

~~prop~~ Let Γ be a fin. group with $o \in \Gamma$ identity. $f : \Gamma \times \Gamma \to [0,\infty]$ is invariant iff $\forall x, y, \gamma \in \Gamma$, $f(x,y) = f(\gamma x, \gamma y)$. Then

$$\underbrace{\sum_{x \in \Gamma} f(o,x)}_{\text{mass out}} = \underbrace{\sum_{x \in \Gamma} f(x,o)}_{\text{mass in}}.$$

Pf. $\sum_{x \in \Gamma} f(o,x) = \sum_{x \in \Gamma} f(x^{-1}, o) = \sum_{x \in \Gamma} f(x, o)$

Example (grandparent graph) Take an infinite path in 3-reg. tree (free product of 3 copies of $\mathbb{Z}_2$). From each vx. $\exists$ unique ray in that direction - connect each vertex to its grandparent - the resulting graph is transitive but the mass transport of $f(x,y) = \begin{cases} 1 & \text{if } y \text{ is } x\text{'s grandparent} \\ 0 & \text{o/w} \end{cases}$ at root isn't 0. - $\sum f(o,x) = 1$, $\sum f(x,o) = 4$. Invariant under graph automorphisms.