W11 D3
6/1/15

Random Spanning Trees

Reminder

If G & G^* are loc. fin., planar & dual to eachother, $F \subseteq E(G)$ is a spanning forest, define $F^* \subseteq E(G^*)$ defined by $e^* \in F^* \iff e \notin F$. [really a forest when F has no finite components - otherwise it's still spanning but may contain a cycle.

Prop $F \sim WMSF(G) \Rightarrow F^* \sim FMSF(G^*)$.

Cor In same setting, if each tree in WMSF of G is one-ended a.s., then the FMSF of G^* in a single tree (boundry of 2 components $\overset{*}{\Rightarrow}$ bi-infinite path).

Invasion Percolasion & MST

For each $x \in V$ define $T(x) = \bigcup_{n \geq 0} t_n$ where $t_0 =$ [empty tree on x], and given t_n, t_{n+1} is $t_n \cup \{\text{smallest edge touching } t_n \text{ on exactly one side}\}$.

Prop. $\bigcup_{x \in V} T(x) = WMSF$ for any U injective.

pf. Recall $e \in WMSF \iff \exists W \subset V$ finite & e is lowest in $\partial_E W$. So if $e \in T(x)$, then there was some n s.t. $e \in t_n \setminus t_{n-1}$, so taking $W = V(t_{n-1})$ would work. If $e \in WMSF$, $\exists$ such W, and $e \in \bigcup_{x \in W} T(x)$ (in fact, for any $x \in \{W \text{ with edges to } V \setminus W\}$, $e \in T(x)$) since this tree must exit W for the first time in e.

Prop MSF on $\mathbb{Z}_2$ is a.s. a tree.

pf. We proved that on $\mathbb{Z}_2$, percolation with $p = 1/2$ has no infinite clusters. So, if $e \in E(\mathbb{Z}^2)$ s.t. $U(e) \leq 1/2$, then both endpts. belong to the same $\underbrace{\text{cluster}}_{\text{of WMSF}}$ by prev. prop. Therefore, if it had 2 components, all edges between them have value $> 1/2$. So, $\{e^* \in {\mathbb{Z}^2}^* \mid u^*(e^*) < 1/2\}$ spans an infinite cluster, which has probability 0 by Harris' thm

Example (Uniqueness not monotone) Let $T =$ binary tree with each edge replaced by a 2-path, $p_c(T) = \sqrt{1/2}$. Take $\mathbb{Z}^2$ and

This won't happen in transitive graphs -

Thm (Häggström Peres X Schramm '99) G transitive, loc. fin. If $p_2 > p_1 > p_c(G)$ & a.s. $\exists!\ \infty$ p_1-cluster then a.s. $\exists!\ \infty$ p_2 cluster.

Simultanous Coupling $U(e)$ are i.i.d. $\sim U[0,1]$, $w_p = \{e \mid U(e) < p\}$. the thm. can be ~~restated as~~ proved by / proving that under the same setting, a.s. any infinite p_2-cluster contains an infinite p_1-cluster (p-cluster = cluster in w_p).

Slight Variation on invasion percolasion

Start at $x \in V$, $I_1(x)$ = lowest edge incident to x, $I_{n+1}(x) = I_n(x) \cup \{e\}$ where e is the lowest edge touching $I_n(x)$ not in $I_n(x)$, $I(x) = \bigcup_n I_n(x)$.

properties

1) Given $p \in [0,1]$ η an ∞ p-cluster, $x \in \eta$ - $I(x) \subseteq \eta$.

2) If $\exists e \notin I_m(x)$ that touches $I(x)$ and $U(e) < p$, $\exists \infty$ p-cluster η s.t. $|I(x) \cap \eta| = \infty$.

Main Thm. G transitive, then a.s. $\forall p > p_c$, all $x \in V$. $\exists$ some ∞ p-cluster intersecting $I(x)$.

Cor $p_2 > p_1 > p_c$ - any inf. p_2-cluster $\supseteq$ inf. p_1 cluster.

pf. η inf. p_2-cluster, take $x \in \eta$ so $I(x) \subseteq \eta$. By main thm. $I(x)$ intersects some infinite p_1 cluster.

Lemma 1 G inf. bounded degree, $x \in V$, $R \in \mathbb{N}$. Then a.s. $I(x)$ contains $B_R(y)$ for some $y \in V$.

edges with $\mathrm{dist}(e, y) < R$.

pf. Let $S_r(x) = \{y \in V \mid \mathrm{dist}(x,y) = r\}$, and

$T_n = \inf\{k \mid \mathrm{dist}_G(I_k(x), S_{2nR}) = R\}$. $|I(x)| = \infty$ so $T_n < \infty$. Let $y_n \in S_{2nR}$ be s.t. $\mathrm{dist}(I_{T_n}(x), y_n) = R$. Let A be the prob. 1 event "$\forall p < p_c$ $\nexists$ ∞ p-cluster". $A_n = \{\forall e \in B_R(y_n)$ $U(e) < p_c(G)\}$ If $A \cap A_n$ occurs, $B_R(y_n) \subseteq I(x)$. Note $B_R(y_n)$

about values in $B_R(y_n)$. Then $P(A_n | y_n, A_1, \dots, A_{n-1}) \geq$
$p_c^{|B_R(y_n)|} \geq p_c^{d^R} > 0$ so it will happen for some n with probability 1.

Def: $\xi_p(x) = \#$ edges (y,z) s.t. $y \in I(x)$ and z in some ∞ p-cluster.

Lemma 2 G transitive, $p > p_c$, $x \in V$. Then a.s. $\xi_p(x) = \infty$.

pf Fix $\varepsilon > 0$. $\exists R$ s.t. $\forall y \in V$, $\mathbb{P}(B_R(y) \cap$ some inf. cluster$) \geq 1-\varepsilon$, since there is one with prob. 1. This can be rephrased - if $F \subseteq E(G)$ finite, $A(F) = \{\exists \infty$ path in ω_p that starts at distance 1 from F and won't visit F again$\}$. So if $B_R(y) \subseteq F$, $\mathbb{P}(A) \geq 1-\varepsilon$.

By lemma 1, a.s. $\exists k$ s.t. $I_k(x)$ contains $B_R(y)$ for some y. let τ be 1^{st} such k. Since invasion up to time τ does not reveal information about edges outside $I_\tau(x)$ we get $\mathbb{P}(A(I_\tau(x)) \mid I_\tau(x)) \geq 1-\varepsilon$, so $\mathbb{P}(A(I_\tau(x))) \geq 1-\varepsilon$. But whenever $A(I_\tau(x))$ holds, $\xi_p \geq 1$ so $\mathbb{P}(\xi_p \geq 1) = 1$. If $\mathbb{P}(\xi_p = n) > 0$ we can do some local changes to get $\mathbb{P}(\xi_n = 0) > 0$. Therefore $\mathbb{P}(\xi_p = \infty) = 1$.