29/10/14
W1 D4

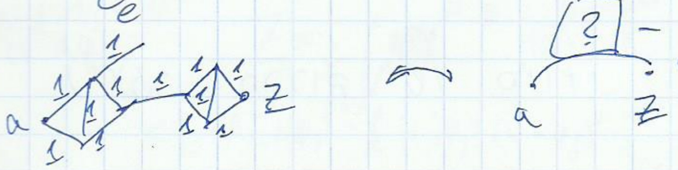
Rand. Spanning Trees

Electric Networks

Doyle & Shell Chapter 8 – Prob. on Trees, an introductory climb (Peres).

Problem $\mathbb{P}$(RSW in $\mathbb{Z}^2$ reaches $(0,1)$ before returning to $(0,0)$) $= ?$ turns out it's $\frac{1}{2}$.

Setting A network is a conn. graph G with non-neg edge weights $\{c_e\}_{e\in E}$ (conductances). Their recipocals $r_e = \frac{1}{c_e}$ are called resistances.



– effective conductance

A function $f: V \to \mathbb{R}$ is harmonic at $x \in V$ if

$$h(x) = \frac{1}{\pi_x}\sum_{y\sim x} c_{xy} h(y), \quad \text{where } \pi_x = \sum_{x\sim y} c_{xy}.$$

Distinguish $a, z \in V$ to be the source and sink of the network.

A funtion $V: G \to \mathbb{R}$ which is harmonic on all vertices except for a, z is called a voltage.

Prop If h is a voltage & $h(a) = h(z) = 0$ than $h \equiv 0$.

Proof We'll show $h \le 0$. O/w, $\max_{v\in V} h > 0$. let $A = \{v\in V \mid h(v) = \max h\}$.
If $x \in A$ then every $y \sim x$ is also in A by harmonicity, so $a, z \in A$ in cont. Similarly $h \ge 0$, so $h \equiv 0$.

Cor. If h_1, h_2 are voltages with equal boundary conditions ($h_1(a) = h_2(a)$, $h_1(z) = h_2(z)$) then $h_1 \equiv h_2$.

Proof $h_1 - h_2$ is a voltage func. with $h(a) = h(z) = 0$.

Existance For $\forall x, y \in \mathbb{R}$, $\exists$ voltage V with $V(a) = x$, $V(z) = y$.

Proof Solve the resp. – $n-2$ vars & $n-2$ nonhomogeanus eqs.
Why must a solution exist? Consider a random walk:
A markov chain with states V and trans. probs.
$\mathbb{P}(X_1 = y \mid X_0 = x) = \frac{c_{xy}}{\pi_x}$ [when $c_e = 1$ for $\forall e \in E$ this is just a simple random walk on G].

To construct a voltage h with $h(a)=0$, $h(z)=1$ take $h(x)$ to be $\mathbb{P}(\text{visited } z \text{ before } x \mid X_0 = x)$. It's easy to see harmonicity, ant to costruct h with arbitrary boundry conditions from that.

Flows [defined on directed edges $\vec{E}$]

Def- A flow from a to z is $\theta : \vec{E} \to \mathbb{R}$ satisfying:

$$\theta(xy) = -\theta(yx) \text{ and } \underbrace{\sum_{y\sim x} \theta(\vec{xy}) = 0}_{\text{Kirchhoff's node law - flow in = flow out}} \text{ for } x \neq a, z.$$

Given a voltage h, the current flow associated with h is $I(\vec{xy}) = [h(y) - h(x)] \cdot c_{xy}$. Easy to see it's a flow.

Bijection between voltages (up to an additive konstant) and current flows. A current flow satisfies the cycle law - if $\vec{e_1}, \dots, \vec{e_n}$ is a cycle, $\sum_{i=1}^{n} r_{e_i} I(\vec{e_i}) = 0$

From now on only consider voltages with $V(a)=0$, $V(z) \geq 0$.

The strength of a flow θ, is $\|\theta\| = \sum_{x\sim a} \theta(a\vec{x})$