W10D3 Random Spanning Trees

30/12/14

Next HW: +1 week. Back to MST:

G finite, $\forall e \in E$ $U(e)$ i.id. uniform on $(0,1)$.

$e \notin$ MST if $\exists$ path P in $G \setminus e$ $x \to y$ with all values $< U(e)$.

When G is infinite:

$e \notin$ FMSF if $\exists$ path $x \to y$ in $G \setminus e$ with all values $< U(e)$.

$e \notin$ WMSF if $\exists$ ex. path $x \to y$ in $G \setminus e$ with all values $< U(e)$.

$Z_f(e) := \inf_{\substack{P: x\to y \\ \text{in } G\setminus e \\ \text{path}}} \max\{U(e') \mid e' \in P\}$, $e \in$ FMSF iff $U(e) < Z_f(e)$.

$Z_f(e)$ - same for ex. path (sup instead of max), $e \in$ WMSF iff $U(e) < Z_w(e)$.

Properties 1) WMSF $\subseteq$ FMSF (trivial coupling.

(all a.s.) 2) No cycles in both.

3) Both spanning.

4) Infinite conn. components.

5) G conn. loc. fin. ameanable the "avg. deg" of WMSF, FMSF is 2.

If V_n exhausting sets s.t. $|\partial V_n| = o(|V_n|)$,

$\lim_n \frac{1}{|V_n|} \sum_{x \in V_n} \deg_{W/FMSF}(x) = 2$ a.s & in expectation.

This was true for every forest s.t. all components are infinite.

6) If both W/FMSF have same fin. num. of components or $\mathbb{E}\deg(x)$ is the same in both $\Rightarrow$ FMSF = WMSF, since WMSF $\subseteq$ FMSF.

7) If G transitive & ameanable WMSF = FMSF.

8) If WMSF is conn. a.s., WMSF = FMSF.

9) If each component of FMSF has one end $\Rightarrow$ WMSF = FMSF.

10) Both invariant under graph automorphisms.

We say G has almost-everywhere-uniqueness if Leb-a.e. $p \in [0,1]$ $\exists$ at most 1 inf. cluster (conn. comp.) in p-percolation on G.

Prop FMSF = WMSF iff G has a.e. uniqueness.

pf. Fix $e \in E$, $p = u(e)$ and p-percolation on $G \setminus e$.

$A_e = \{$two end points of e are in saperate infinite components of $(G\setminus e)_p\}$. Then $A_e = \{e \in \text{FMSF} \setminus \text{WMSF}\}$, so WMSF = FMSF $\Leftrightarrow P(A_e) = 0$ $(\forall e \in E)$. Let $f(e,p) = P(\text{end pts of } e \text{ are in } 2\ \infty \text{ clusters of } (G\setminus e)_p)$.

$\mathbb{P}(A_e \mid u(e)) = f(e, u(e))$. So, $P(A_e) = \int_0^1 f(e,p)dp$. Let $B(e) \subseteq [0,1]$ the set of pts. p s.t. $f(e,p) > 0$.

$P(A_e) = 0 \Leftrightarrow \text{Leb}(B(e)) = 0$, $\forall e$ $P(A_e) = 0 \Leftrightarrow \text{Leb}(\bigcup_{e \in E} B(e)) = 0$

The first is WMSF = FMSF and 2nd is a.e. uniqueness.

Transitive Graphs ameanable $\Rightarrow$ set of non-uniqueness $= \emptyset$

Question #2 eq. to non-ameanable $\Rightarrow$ set of non-uniqueness = interval.

Thm G transitive, conn. N_∞ = # inf. clustents in p-perc, Then N_∞ is a.s. constant $\in \{0, 1, \infty\}$.

pf. Enough to show $\mathbb{P}(N_\infty \in \{0,1,\infty\}) = 1$ since $\{N_\infty = 0\}$ and $\{N_\infty = \infty\}$ are tail events. N_∞ measurable & invariant to Γ the group of isomorphism. Let A be a Γ-invariant event, and we'll prove $\mathbb{P}(A) \in \{0,1\}$.

$\exists$ ~~finite set of edges~~ event B depending on finitely many edges s.t. $P(A) \overset{\varepsilon}{\approx} \mathbb{P}(B)$. Take $\gamma \in \Gamma$ such that γB and B are independet. $\mathbb{P}(\gamma A) = \mathbb{P}(A) \overset{\varepsilon}{\approx} \mathbb{P}(\gamma B)$, so $\mathbb{P}(A) = \mathbb{P}(\gamma A \cap A) \overset{\varepsilon}{\approx} \mathbb{P}(\gamma B \cap B) \overset{\varepsilon}{\approx} \mathbb{P}^2(A)$. Take $\varepsilon \downarrow 0$ to get $\mathbb{P}(A) = \{0,1\}$. The result now follows from:

Insertion/Deletion tolerance

Given $w \in \{0,1\}^{E(G)}$ and $e \in E(G)$ put $\Pi_e w = w \cup \{e\}$. If $A \subseteq \{0,1\}^{E(G)}$ define $\Pi_e A = \{\Pi_e w \mid w \in A\}$. Then, in p-percolation $\mathbb{P}(\Pi_e A) \geq p\,\mathbb{P}(A)$ (similarly for removing e, replacing p with $1-p$).

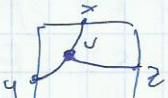
(Burton-Keane 91) Thm G transitive & amenable $\Rightarrow \exists$ at most 1 ∞-cluster for $\forall p$.

Lemma Let T be a tree with all degrees >1, $B = \{$vertices of degree $\geq 3\}$. Then, for any finite $K \subseteq V(T)$, $|\partial_E K| \geq |K \cap B| + 2$.

pf. By induction on $|K|$. $K=1$ trivial to make sure. For general K, $K \cap T$ fin. tree - so $\exists$ vertex v of deg. 1. Let $K' = K \setminus v$. By ind. hyp. $|\partial K'| \geq |K' \cap B| + 2$. If $\deg v = 2$ both sides won't change, if 3 both increase by 1, o/w RHS ~~stays the same~~ increases but LHS increases, even more.

$C(x) :=$ component of x in p-percolation.
$B_R(x) :=$ set of edges in G with both endpts at distance $\leq R$ to x.
x is a furcation if removing it splits $C(x)$ to ≥ 3 clusters.

pf of thm. Assume by contradiction $\mathbb{P}(N_\infty = \infty) = 1$. Let $\Lambda = \{$furcation pts$\}$. We'll prove (1) $c = \mathbb{P}(x \in \Lambda) > 0$ but (2) $\forall K \subset V$ finite $|\partial_E K| \geq c|K|$, contradicting amenability.

(1) Is just done with prob. >0 by taking a large ball that intersects 3 inf. disjoint components. So, $\exists x, y, z$ on the boundry s.t. $\exists$ inf. paths from each of them to infinity. By independence $\exists$ prob >0 that the inside of the ball is  and then v is a furcation point with probability >0.

(2) $\forall K \subseteq V$, $\mathbb{E}[|K \cap \Lambda|] = c|K|$. We claim $|\partial_E K| \geq K \cap \Lambda$. Let $T(\eta)$ be a spanning tree of an ∞ cluster η. Remove vertices of degree 1 and iterate call the resulting tree $T'(\eta)$. Every furcation point won't be removed and has $\deg \geq 3$ in $T'(\eta)$. So by Lemma $|\partial_E K \cap T'(\eta)| = |\partial_{E(T(\eta))} K \cap T'(\eta)| \geq |K \cap \Lambda \cap T'(\eta)|$. Sum over η's - the LHS $\leq |\partial_E K|$ but RHS is $|K \cap \Lambda|$.