W9D4 **Random Spanning Trees**

24/12/14

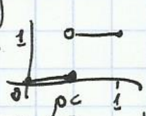
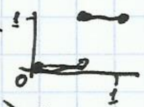
Introduction to percolation

G infinite, connected, loc. fin..

$\{U(e)\}_{e\in E}$ uniform i.i.d. on $(0,1)$.

Given $0\le p\le 1$ define $G_p=\{e \mid U(e)\le p\}$. Note $p_1<p_2 \to G_{p_1}\subseteq G_{p_2}$

$f(p)=\mathbb{P}_p(\exists \text{ inf. conn. comp.})\in\{0,1\}$ by Kolmogorov's 0-1 law.

$f(p)$ non-decreasing, $f(0)=0$, $f(1)=1$. Either  or .

In particular $\exists p_c\in[0,1]$ s.t. $\forall p<p_c$ $f(p)=0$, $\forall p>p_c$ $f(p)=1$.

Examples:

On $\mathbb{Z}$, $p_c=1$ ($f(p_c)=1$). On a d-reg. tree ($d\ge 2$) $p_c=\frac{1}{d-1}$ ($f(p_c)=0$, branching process).

Thm. in $\mathbb{Z}^2$ $f(1/2)=0$ (so $p_c\ge 1/2$). (Harris 60)

(Kesten 80) $p_c=1/2$.

#1 Open Question - p_c of $\mathbb{Z}^3$? What is $f(p_c)$.

Known: $p_c(\mathbb{Z}^3)<p_c(\mathbb{Z}^2)$.

Conj. For any trans. graph s.t. $p_c<1$, $f(p_c)=0$.

Known for: $\mathbb{Z}^2$, $\mathbb{Z}^d$ $d\ge 15$, G non-ameanable Cayley graph.

Uniqueness: When $p>p_c$ how many ∞ clusters are there?

Thm. (Burton-Keane '91) G transitive & ameanable $\Rightarrow \forall p$ G_p has a.s. ≤ 1 inf. comp.

Q #2 (Benjamini-Schramm) Does G transitive non-ameanable $\Rightarrow \exists p>p_c$. G_p a.s. has >1 inf. comp.

The event $\{\exists \text{ unique } \infty \text{ comp.}\}$ isn't generally mon. $g(p)=\mathbb{P}_p(\exists! \text{ inf. comp})\in\{0,1\}$.

Thm it is for G transitive. $p_u=\inf\{p \mid g(p)=1\}=\sup\{p \mid g(p)=0\}$.

Q #2 rephrased: Does G non-ameanable $\Leftrightarrow p_c(G)<p_u(G)$.

Example $T_d\times\mathbb{Z}$ for $d\ge 100$, $p_c<p_u<1$.

Thm. $\forall$ trans. G & $p<p_c$, $\mathbb{E}(|C(p)|)<\infty$. Exp decreasing follows. $\mathbb{P}(|C(p)|\ge n)\le e^{-cn}$. What if $p=p_c$?

Thm. In tri. lattice, $\mathbb{P}_{p_c}(|C(p)|\ge n)\simeq n^{-91/48}$ ($C(p)$: comp. containing ρ). $G=\mathbb{Z}^2$ open.

(under certain assumptions)