W9D3

23/12/14

Random Spanning Trees

Minimal Spanning Trees (& Percolations)

Let $G=(V,E)$ be a fin. conn. graph. Define random variables $\{u(e) \mid e \in G\}$ i.i.d. uniform on $(0,1)$.

e is lower then e' if $u(e) < u(e')$. (sim. higher, lowest, highest)

Let T_u be graph on V and $e \notin E(T_u)$ iff $\exists$ some other path ~~containing~~ connecting e^- and e^+ ~~contains some e' such that e is lower than e'~~ with all edges lower then e.

T_u has no cycles - in each cycle, T_u can't contain the highest edge of the cycle.

T_u is connected since $\forall A \subseteq V$, the lowest edge between $A, V \setminus A$ is in T_u.

Observe T_u has the lowest edge label $\underbrace{\sum_{e \in T} u(e)}_{\text{sum}}$, among all spanning trees.

Indeed, let t' be another spanning tree. Let e be the lowest in $t' \setminus T_u$. Then, replacing e with the lowest edge connecting e^-, e^+ in T_u decreases this sum. Therefore the minimum is achieved for T_u.

Prop. Let G, H conn. fin. graphs T_G, T_H MSTs as above.

If $G \subseteq H$ then T_G dominates $T_H \cap E(G)$.

If G is obtained from H by identifying vertices, then T_G is dominated by $T_H \cap E(G)$.

pf. Just by definitions...

On infinite Graphs

G inf. conn. loc. fin., $U: E \to \mathbb{R}^+$ injective. The free minimal spanning forest $F_f(U)$ (FMST) is the set of edges (x,y) such that any path $x \mapsto y$ has a higher edge then (x,y).

Def. an extended path $x \to y$ is either a path $x \to y$ or two disjoint rays from x and from y.

The <u>WMST</u> is the set of edges $\mathcal{F}_w(U) = \{(x,y)$ s.t. for all <u>extended</u> paths $x \to y$, some e' is higher than $e\}$.

<u>Prop</u> 1) $\mathcal{F}_w \subseteq \mathcal{F}_F$.

2) no cycles in both.

3) Both are spanning.

4) All conn. comps. are infinite.

<u>Exercise</u> FMST is the lim. of MST on G_n $\left(\begin{array}{c} G_n \text{ fin. exhaustion} \\ \text{of } G \end{array}\right)$.

WMST " " " MST on G_n^w

<u>The dual Definition</u>

$\mathcal{F}_w(U)$ [F] $= \{$edges e s.t. $\forall A \subseteq V$ finite [fin./inf.] s.t. e is lowest edge connecting A to $V \setminus A\}$.

If e is such, then clearly $e \in$ WMST (any other path uses another edge from the cut)

If $e \in$ WMSF then the finite among "vertices connected to x/y using lower edges" works as A.

<u>Define</u> $Z_F^u(e) = \inf\limits_{\substack{P:\text{ path} \\ e_- \to e_+}} \max\{U(e') : e' \in P\}$, then $\mathcal{F}_F(U) = \{e \mid U(e) \leq Z_F(e)\}$. Similarly for $Z_w^u(e) = \inf\limits_{\substack{P\text{ ex. path} \\ e_- \to e_+}} \sup\{U(e') \mid e' \in P\}$,

$\{e \mid U(e) < Z_w(e)\} \subseteq \mathcal{F}_w(U) \subseteq \{e \mid U(e) \leq Z_w(e)\}$. In our case $U(e)$ i.i.d. uniform so those sets are all equal.

<u>Def</u> If $W \subseteq V$ $\partial_E W = \{$edges $W \leftrightarrow V \setminus W\}$.

<u>Lemma</u> If $U : E \to \mathbb{R}$ injective, $Z_F(e) = \sup\limits_{e \in \Pi \text{ cut}} \inf\{U(e') \mid e' \in \Pi \setminus e\}$ and $Z_w(e)$ is the same with the additional requirement that Π is finite (so the sup becomes a max).

<u>Prop.</u> U as before, $\forall p < 1$ edges with $U(e) < p Z_w^u(e)$ are subtrees of $\mathcal{F}_w(U)$ with at most one end. $\forall \eta > 0$ edges with $U(e) \leq (1+\eta) Z_{F}^u(e)$ $\forall e \in V$ are supergraphs of $\mathcal{F}_F(U)$ (conn.).

<u>pf.</u> Let ξ_p be those edges. P bi-inf. path, $\alpha = \sup\limits_{e \in P} U(e)$. If $P \subseteq \xi_p$ then $U(e) \leq p \cdot \alpha \Rightarrow \alpha \leq p\, \alpha$ (take sup). For η, Let ξ_η be those edges, Π cut, $\varphi = \inf\limits_{e \in \Pi \text{ cut}} U(e)$, $\exists e \in \Pi$ $U(e) \leq \varphi(1+\eta)$ ($\leq Z_F(e)$) so $e \in \xi_\eta \Rightarrow \xi_\eta$ conn.