W8D4

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Random Spanning Trees

Lemma 1 $\lim_{N\to\infty} \inf\{C^w_{eff}(K \leftrightarrow \infty; \mathbb{Z}^d \setminus B(n)) \mid K \subseteq \mathbb{Z}^d \setminus B(n), |K| = N\} = \infty$

proved last time for $\mathbb{Z}^d$, it's clear how to do for $\mathbb{N} \times \mathbb{Z}^{d-1}$, and then for $\mathbb{Z}^d \setminus B(n)$.

Lemma 2 G is a network, $L \subseteq E(G)$ finite, $\forall e \in L$ write $\hat{r}_e = R^w_{eff}(e^- \leftrightarrow e^+; G \setminus L)$. Then $\mu^w(L \cap F = \emptyset) \geq \geq \prod_{e\in L} \frac{1}{1 + c(e)\hat{r}_e}$.

pf. By induction. $\mu^w(e \in F) = c(e) \cdot R^w_{eff}(e^- \leftrightarrow e^+; G) = \frac{c(e)}{c(e) + C^w_{eff}(e^- \leftrightarrow e^+; G \setminus e)} \leq$ (R. Mon.) $\frac{c(e)}{c(e) + C^w_{eff}(e^+ \leftrightarrow e^-; G \setminus L)} \Rightarrow \mu^w(e \notin F) \geq 1 - \frac{1}{1 + \frac{C^w_{eff}(e^- \leftrightarrow e^+; G \setminus L)}{c^w(e)}} = \frac{1}{1 + \frac{c(e)}{C^w_{eff}(e)}} =$

$= \frac{1}{1 + \hat{r}_e c(e)}$.

Now $\mathbb{P}(e_1, e_2 \notin F) = \mathbb{P}(e_1 \notin F)\,\mathbb{P}(e_2 \notin F \text{ of } G \setminus e_1)$ and continue the same way.

Thm. μ^w a.s. each tree of F is one-ended.

pf. We proved for $d=2$. If $d \geq 3$ ($\mathbb{Z}^d$ transient). Let A_e be the event that $A \setminus e$ has a finite component. Suffices to show $\mu^w(A_e \mid e \in F) = 1$.

Exhaust $\mathbb{Z}^d$ by boxes centered at e. Let $\mathcal{F}_n = \sigma(\{f \in F\} : f \in E(B(n)) \setminus e)$. G_n is $\mathbb{Z}^d$ after deleting edges of $E(B(n)) \setminus e$ which aren't in the tree and contract those which are. WLOG $e \in G_n$.

By Kirchhoff $\mu^w[e \in F \mid \mathcal{F}_n] = R^w_{eff}(e^- \leftrightarrow e^+; G_n) = \frac{1}{1 + C^w_{eff}(e^- \leftrightarrow e^+; G_n \setminus e)}$.

By Levý 0-1 $\mu^w[e \in F \mid \mathcal{F}_n] \to \mu^w[e \in F \mid \mathcal{F}_\infty]$. For almost every $w \in \{e \in F\}$ we have $\mu^w[e \in F \mid \mathcal{F}_\infty](w) > 0$, because if $B = \{w : \mu^w[e \in F \mid \mathcal{F}_\infty](w) = 0\} \in \mathcal{F}_\infty$, $\mu^w(B \cap \{e \in F\}) = 0$. So for a.e. $w \in \{e \in F\}$

$C^w_{eff}(e^- \leftrightarrow e^+; G_n \setminus e) \leq C(w) < 0$.

So, by lemma 1 and the triangle ineq., $R^{w}_{eff}(x \leftrightarrow y) \leq R_{eff}(x \leftrightarrow \infty) + R_{eff}(y \leftrightarrow \infty)$. So for a.e. $w \in \{e \in F\}$

$\exists N(w)$ s.t. $\forall n \geq n$ at least one of ~~[illegible]~~ $\deg(e^-), \deg(e^+)$ in G_n is at most N. Assume WLOG that it's $\deg(e^-)$. Let L be those edges from e^- in G_n - $N \cdot \mathbb{Z}^{d-1}$ is transient, $\hat{r}_f \leq C$ for all $f \in L$. By lemma 2, $\mu^w(A_e \mid F_n) \geq \left(\frac{1}{1+C}\right)^{N(w)}$ for a.e. w. By Levy's 0-1 law,

$\mu^w[A_e \mid F_n] \to \mathbb{1}_{A_e}$ so it must converge to 1.