W3 D#3
16/12/14

Random Spanning Trees

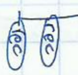
Geometry of Trees in the WSF

Remark from last time: $r_N = \sum_{n,m} \mathbb{1}_{(x_m = Y_n)} \chi(m,n)$ has infinite exp. $\rightarrow$ $|\mathbb{E}(X) \cap Y| = \infty$ requires transitivity.

Thm. μ^W-a.s. all trees T have the property: μ^W-a.s. the WSF on T is all of T.

In particular, if c is bounded, T is recurrent (Q5, ex. 2).

Observation If (T, c) (c bounded) is a transient network for a tree T, then $\exists e \in T$ s.t. both components of $T \setminus e$ are transient.

pf. Otherwise $\forall e \in T$, T/e has 1 recurrent comp. It follows that the tree has the form  ... but the line is also recurrent so the whole graph is.

pf. of thm. For any $e \in E$ define $A_e = \{$both endpoints of e are in transient comp. of $F\}$. Our goal - $\mu^W(e \in F, A_e) = 0$ [assume c bdd. $\Rightarrow$ this shows T is recurrent by the obs.].

Enumerate $G \setminus e$ by $e_1, e_2, \ldots$ and let $\mathcal{F}_n$ be the σ-field $\sigma(\{f \in F\}, f = e_1, \ldots, e_n)$. Let G_n be the graph where we contract $\{e_1, \ldots e_n\} \cap F$ and erase $\{e_1, \ldots, e_n\} \cap F$.

By Kirchhoff $\mu^W(e \in F | \mathcal{F}_n) = i(e, G_n) = c_e R(e^- \leftrightarrow e^+; G_n)$.

We claim that on A_e, this converges to 0 a.s., i.e. $\mu^W(e \in F | \mathcal{F}_n) \mathbb{1}_A \xrightarrow[n \to \infty]{a.s.} 0$. Given the claim, this limit is

$\mu^W(e \in F | \mathcal{F}_\infty) \mathbb{1}_{A_e} = 0 \Rightarrow \mathbb{P}(e \in F, A_e) = 0$.

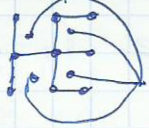
proof of the claim: if H is a transient network, $x \in V$, and $\{e_1, \ldots\} = E(H)$, $H_n = H / \{e_1, \ldots, e_n\}$. If θ is a UCF $x \to \infty$, $\forall \varepsilon \exists N$ $\sum_{i \geq N} \theta(e_i)^2 < \varepsilon$. Project θ to $H_n \Rightarrow$ an $x \to \infty$ flow in H_n, $\mathcal{E}(\theta, H_n) \leq \varepsilon$. So $R(e^- \leftrightarrow \infty; G_n) \xrightarrow{n \to \infty} 0$ on A_e, and similarly for e^+ (both are μ^W-a.s.).

By the triangle ineq., $R_{eff}(e^- \leftrightarrow e^+; G_n) \to 0$ on A_e.

An infinite simple path $v \to \infty$ in a tree is a ray. 2 rays are eq. if $|V_1 \Delta V_2| < \infty$. An "end" is an equivalence class.

Thm. On $\mathbb{Z}^d$ $d \geq 2$, each component of WSF has one end

For a fin. planar graph $\exists$ bij. between spanning trees of G, G^* — $e \in T \Leftrightarrow e^* \notin T^*$.



If $c(e)$ are conds. of G, put $c(e^*) = r(e)$, so that the weights $w(T) = \frac{w(T^*)}{\prod_{e^* \in G^*} c(e^*)}$, $w(T) \sim w(T^*)$.

If G planar & infinite (locally finite $\to$ so is G^*), FSF on G is dual to WSF on G^*. On $\mathbb{Z}^2$ [FSF=WSF] the dual is the graph itself, so $\mathbb{P}(e \in T) = \mathbb{P}(e^* \in T^*) = \frac{1}{2}$.

Thm. If G is simple, planar, recurrent, G^* locally fin. & recurrent, the UST on G has 1 end.

pf. recurrent $\Rightarrow$ single tree in WSF/FSF. T has a bi-inf. path $\Rightarrow$ T^* has ≥ 2 components.

Lemma 1: Let $B(n)$ be a box of side length n in $\mathbb{Z}^d$, $d \geq 3$, then $\lim_{N \to \infty} \sup \{ R_{eff}(K \leftrightarrow \infty; \mathbb{Z}^d \setminus B(n)) : K \subseteq \mathbb{Z}^d \setminus B(n), N = |K|, n \geq 0 \} = 0$.

pf. We'll show $\lim_{N \to \infty} \sup \{ R_{eff}(K \leftrightarrow ; \mathbb{Z}^d) : |K| = N \} = 0$

later $\mathbb{N} \times \mathbb{Z}^{d-1}$

$G = \mathbb{Z}^d$ is trans. Let θ be a UCF $0 \to \infty$ with fin. $\mathcal{E}(\theta) = c$. $\forall \varepsilon \, \exists l \sum_{e \in E(G), d(e,0) \geq l} \theta^2(e) < \varepsilon$. Let θ_x be the same flow $x \to \infty$, $K = \{x_1, \dots, x_N\}$.

$\exists$ at least $M = \frac{N}{(4l)^d}$ pts. in K of pairwise dist $\geq l$, call them $x_1, \dots, x_M$. Take $\theta = \frac{1}{M} \sum_{i=1}^{M} \theta_i$. Then

$\mathcal{E}(\theta) = \langle \theta, \theta \rangle = \frac{1}{M^2} M c + \dots$ If $i \neq j$, $\langle \theta_{x_i}, \theta_{x_j} \rangle = \sum_{e, d(e, x_i) \leq l} \theta_{x_i}(e) \theta_{x_j}(e) + \dots \leq$ $2\sqrt{\varepsilon}$. So, $\mathcal{E}(\theta) = \frac{c}{M} + 2\sqrt{\varepsilon} \cdot c$.