W7 D4
10/12/14

Random Spanning Trees

Lemma 2 Assume $a_N = \sum_z G_N(z,\rho)^2 \longrightarrow \infty$. Then $\forall x_0, y_0$,

$$\liminf_{N\to\infty} \frac{(\mathbb{E}_{x_0,y_0} I_N)^2}{\mathbb{E}_{x_0,y_0} I_N^2} \geq \frac{1}{16}.$$

pf. We've shown last time $\mathbb{E}_{x_0,y_0} I_N^2 \leq 16\, a_N^2$, & $\mathbb{E}_{x_0,y_0} I_N \leq a_N$.

$u_N(x,y) = \frac{\mathbb{E}_{x,y} I_N}{a_N} \leq 1$, $u(x,x) = 1$. Fix some r s.t. $\mathbb{P}^r(x_0,y_0) > 0$.

$$a_{r+N} = \sum_{m,n \leq r+N} \mathbb{P}_{x_0,x_0}(X_m = Y_n) = \overbrace{\sum_{m,n\leq N} \mathbb{P}_{x_0,x_0}(X_m = Y_{r+n})}^{I} + \overbrace{\sum_{\substack{m\leq r+N \\ n<r}} \mathbb{P}_{x_0,x_0}(X_m = Y_n)}^{II} + \overbrace{\sum_{m\leq r}\sum_{n\leq N} \mathbb{P}_{x_0,x_0}(X_{N+m} = Y_{r+n})}^{III}.$$

$$I = \mathbb{E}_{x_0,x_0}[a_N u_N(x_0,y_r)] \leq a_N[\mathbb{P}^r(x_0,y_0)u_N(x_0,y_0) + 1 - \mathbb{P}^r(x_0,y_0)].$$

$$II = \sum_{z\in V}\sum_{\substack{m\leq r+N \\ n<r}} \mathbb{P}_{x_0}(X_m = z)\,\mathbb{P}_{x_0}(Y_n = z) = \sum_z G_{r+N}(x_0,z)G_r(x_0,z) \overset{CS}{\leq} \sqrt{a_{N+r}\,a_r}$$

$III \leq \sqrt{a_r a_N}$ similarly. We get

$$\underset{\underset{1}{\wedge}\ a\to\infty}{1} \leq \frac{a_N}{a_{r+N}}[\mathbb{P}^r(x_0,y_0)u_N(x_0,y_0) + 1 - \mathbb{P}^r(x_0,y_0)] + \underset{\downarrow\, 0}{\sqrt{\frac{a_r}{a_{r+N}}}} + \underset{\downarrow\, 0}{\sqrt{\frac{a_r a_N}{a_{r+N}}}} \quad \text{so}$$

we get $u_N(x_0,y_0) = 1 - o_N(1)$.

Cor. $\exists c>0$ $\mathbb{P}_{x_0,y_0}(|(m,n) \text{ s.t. } X_m = Y_n| = \infty) \geq c$.

$\underset{\shortparallel}{I}$

$$\lim_{M\to\infty} \mathbb{P}_{x_0,y_0}(|(m,n) \text{ as above}| \geq M). \text{ Take } N \text{ s.t. } \mathbb{E}_{x_0,y_0} I_N >> M, \text{ so}$$

$$\lim_{M\to\infty} \mathbb{P}_{x_0,y_0}(I_N \geq M) \geq \frac{1}{16}.$$

So, by Levy's 0-1 law, $\mathbb{P}_{x_0,y_0}(I=\infty \mid X_1,\dots,X_n,Y_1,\dots,Y_n) \xrightarrow{a.s.} \mathbb{1}_{(I=\infty)}$.

$\shortparallel$

$\mathbb{P}_{x_n,y_n}(I=\infty) \geq 1/16$ so it's 1 a.s.

Lemma 3 Fix $k>0$ and a path $\{x_j\}_{j=-k}^{0}$ in G. Let $\{X_m\}_{m=0}^{\infty}$ $\{Y_n\}_{n=0}^{\infty}$ be indep. SRW. Set $X_j = x_j$, $j=-k,\dots,0$. Assume $\mathbb{E} I_n \to \infty$. Then $\mathbb{P}(|LE(\{X_m\}_{m=-k}^{\infty}) \cap \{Y_n\}_{n=-k}^{\infty}|) = \infty$

pf. Denote $\{L_j^{(m)}\}_{j=0}^{J(m)} = LE(X_{-k},\dots,X_m)$.

When $X_m = Y_n$, we want to check which configuration of X_m / Y_n hits earlier. On the event $X_m = Y_n$ define $j(m,n) = \min\{j\geq 0 : L_j^{(m)} \in \{X_m, X_{m+1},\dots\}\}$. $i(m,n) = \min(i\geq 0 : L_i^{(m)} \in \{Y_n,\dots\})$.

If $X_m \neq Y_n$ define $i(m,n) = j(m,n) = 0$.

Note If $X_m = Y_n$ & $i(m,n) \leq j(m,n)$, then $L^{(m)}_{i(m,n)} \in LE(\{X_m\}_{n=-k}^{\infty}) \cap \{Y_n\}_{n=0}^{\infty}$

Put $\mathcal{X}(m,n) = \mathbb{1}_{(i(m,n) \leq j(m,n))}$. $\mathbb{E}(\mathcal{X}(m,n) \mid X_m = Y_n = z) \geq 1/2$ since X, Y are interchangable from that point onward.

Write $\gamma_N = \sum_{m,n=0}^{N} \mathbb{1}_{(X_m = Y_n)} \mathcal{X}(m,n)$. Then $\mathbb{E}\gamma_n \geq \sum_{m,n=0}^{N} \frac{1}{2} P(X_m = Y_n) = \frac{1}{2} \mathbb{E}_{x_0, y_0} I_N$.

And note that $\gamma_N \leq I_N$, so $\frac{1}{\mathbb{E}\gamma_N^2} \geq \frac{1}{\mathbb{E} I_N^2}$ and therefore

$\mathbb{P}(\gamma_n \geq \varepsilon E_N) \geq c > 0 \Rightarrow \mathbb{P}(|LE(\{X_m\}_{m=-k}^{\infty}) \cap \{Y_n\}_{n=0}^{\infty}| = \infty) \geq c > 0$.

Pf. of Thm. Let $\Lambda = \{|LE(\{X_m\}_{m \geq 0}) \cap \{Y_n\}_{n=0}^{\infty}| = \infty\}$. By Levy's 0-1 law $\lim_{k \to \infty} \mathbb{P}_{x_0, y_0}\left(\Lambda \,\middle|\, \begin{smallmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{smallmatrix}\right) = \mathbb{1}_{\Lambda}$. But this equals $\mathbb{P}_{x_0, y_k}(\Lambda \mid x_1, \dots, x_k) \underset{\text{lemma 3}}{\geq} c > 0$. Therefore $\mathbb{P}_{x_0, y_0}(\Lambda) = 1$.