W7D3
9/12/14

Uniform Spanning Trees

Recall $G(x,y) = \mathbb{E}_x[\#$ visits to $y]$ - only meaningful on a transient graph. Similarly, $G_N(x,y) = \mathbb{E}_x[\#$ visits before time $N]$

Goal In $\mathbb{Z}^d$, $\#\mu^w$-trees is a.s. $\begin{cases} \overset{\mu^w}{1} & d = \underbrace{1,2,3,4}_{\text{known already}} \\ \infty & d > 4 \end{cases}$

Thm. Let G be transitive and transient.

If $\sum_z G(\rho,z)^2 = \infty$ then 1. $\mathbb{P}_{x_0,y_0}(|\{x_n\}\cap\{y_n\}| = \infty) = 1$.

2. $\mathbb{P}_{x_0,y_0}(|LE(\{x_n\})\cap\{y_n\}| = \infty) = 1$.

3. μ^w-a.s. a single tree.

If $\sum_z G(\rho,z)^2 < \infty$ then 1. $\mathbb{P}_{x_0,y_0}(|\{x_n\}\cap\{y_n\}| = \infty) = 0$ ...... but we don't know this yet

2. μ^w-a.s. has ∞ many trees. this follows

$$\mathbb{E}_{x,x}(|\{X_n\}\cap\{Y_n\}|) = \sum_z \sum_{n,m=1} \mathbb{P}_x(X_n = z)\mathbb{P}_x(Y_n = z) = \sum_z G(x,z)^2.$$

$p^n_1(0,0)$

In $\mathbb{Z}^d$: $d=1$ $P_0(X_{2n}=0) = \dfrac{\binom{2n}{n}}{2^{2n}} \asymp \dfrac{1}{\sqrt{n}}$. In general,

$$\forall d > 1 \quad p^{2n}(0,0) \asymp \frac{1}{n^{d/2}}$$

$$\sum_z G(\rho,z)^2 = \sum_z \sum_{m,n} \mathbb{P}_\rho(X_m = z)\mathbb{P}_\rho(Y_n = z) = \sum_{n,m} p^{n+m}(\rho,\rho) \asymp 2\sum_n \sum_{m=1}^n \frac{1}{(n+m)^{d/2}} \asymp$$

$$\sum_n \frac{1}{n^{d/2-1}} = \begin{cases} \infty & d = 1,2,3,4 \\ <\infty & d > 4 \end{cases}$$

Claim:
When $\sum_z G(\rho,z)^2 < \infty$, $\forall \varepsilon\ \exists x,y \in G$ $\mathbb{P}_{x,y}(|\{x_n\}\cap\{y_m\}| = 0) > 1-\varepsilon$.

pf. We'll use the fact that for G transitive, $p^{2n}(x,y) \leq p^{2n}(x,x)$.

$\mathbb{P}_{x,x}(\exists N\ \forall m,n > N,\ X_m \neq Y_n) = 1 \Rightarrow \exists N\ \mathbb{P}_{x,x}(\forall m,n \geq N\ X_m \neq Y_n) \geq 1-\varepsilon$.

$\mathbb{P}_{x,x}(\forall m,n \geq N,\ X_m \neq Y_n) = \sum_{x,y} \mathbb{P}_{0,0}\binom{X_N = x}{X_N = y}\mathbb{P}_{x,y}\binom{\forall n,m \geq 0}{X_m \neq Y_n} \geq 1-\varepsilon$. So

$\exists x,y,\ \mathbb{P}_{x,y}(\forall n,m \geq 0,\ x_n \neq y_m) \geq 1-\varepsilon$.

So, if you start Wilson's alg rooted at ∞ for WSF, $\mathbb{P}\binom{x,y \text{ are in}}{\text{different comp.}} \geq 1-\varepsilon$.

This is true for $\forall\varepsilon$ so $\mu^w(1 \text{ comp.}) = 0$. Similarly, $\mu^w(2 \text{ comp.}) = 0$

[find $x_1,\dots,x_k$ s.t. $\mathbb{P}_{x_1,\dots,x_k}(\text{No intersection}) \geq 1-\varepsilon$] so μ^w-a.s. we have infinitely many components.

Write $I_N = \sum_{m,n=0}^N \mathbb{1}(x_m = y_n)$. $\mathbb{E}_{x,x} I_N = \sum_z G(x,z)^2 \to \infty$ We'll use this argument:

Lemma 1 $\dfrac{(\mathbb{E}_{x,x} I_N)^2}{\mathbb{E}_{x,x} I_N^2} \geq \dfrac{1}{4}$, Lemma 2 $\forall x_0,y_0 \liminf_{N\to\infty} \dfrac{(\mathbb{E}_{x,y} I_N)^2}{\mathbb{E}_{x,y} I_N^2} \geq \dfrac{1}{16}$. Then we'll use

Levi's 0-1 law to prove the main thm.

Prop. $Z \geq 0$ r.v. $\forall \varepsilon$ $\mathbb{P}(Z \geq \varepsilon \mathbb{E} Z) \geq (1-\varepsilon)^2 \frac{(\mathbb{E} Z)^2}{\mathbb{E} Z^2}$.

pf. Let A denote this event.

$\mathbb{E}[Z \mathbb{1}_A]^2 \leq \mathbb{E}[Z^2]\, \mathbb{P}(A)$, but $(\mathbb{E}[Z\mathbb{1}_A])^2 = [\mathbb{E}Z - \mathbb{E}Z_{A^c}]^2 \geq$ $(1-\varepsilon)^2 (\mathbb{E} Z)^2$.

Lemma 3 Will be specified later...

Levy 0-1 law – Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a prob. space, $\{F_i\}_{i\geq 0}$ a filtration, $F_\infty = \sigma\left(\bigcup_{i=0}^{\infty} F_i\right)$. Let X be an r.v. in L^1. Then $\mathbb{E}[X|F_k] \xrightarrow[a.s.\; L^1]{k\to\infty} \mathbb{E}[X|F_\infty]$. If $X = \mathbb{1}_A$, $F_\infty = F$, $\mathbb{P}(A|F_k) \to \mathbb{1}_A$. Most frequently, we use that in bounding $\mathbb{P}(A|F_k)$ and get that it can't be too small (resp. large) $\mathbb{P}(A) = 1$ (resp. 0).

Lemma 1 pf. $\mathbb{E}_{0,0} I_N = \sum_z G_N(0,z)^2$. $\mathbb{E}_{0,0} I_N^2 = \sum_{z,w} \sum_{m,n=0}^{N} \sum_{j,k=0}^{N} \mathbb{P}_{0,0}(x_m = z = y_n \,\&\, x_j = w = y_k) =$

$$\underbrace{\sum_{z,w} \sum_{m,j}^{N} \mathbb{P}_0(x_m = z, x_j = w) \sum_{n,k}^{N} \mathbb{P}_0(y_n = z, y_k = z)}_{=} .$$

$$\mathbb{P}_0(x_m = z, x_j = w)\, \mathbb{P}_0(y_n = z, y_k = w)$$

$\sum_{m,j}^{N} \mathbb{P}_0(x_m = z, x_j = w) = \sum_{m\leq j} \mathbb{P}_0(x_m = z)\, \mathbb{P}_0(x_j = w | x_m = z) + \sum_{m > j} \mathbb{P}_0(x_j = w)\, \mathbb{P}_0(x_m = z | x_j = w) \leq$

$\leq G_N(0,z) G_N(z,w) + G_N(0,w) G_N(w,z)$. So, the large sum is at most $\mathbb{E}_{0,0} I_N^2 \leq \sum_{z,w} (G_N(0,z) G_N(z,w) + G_N(0,w) G_N(w,z))^2 \leq 4 \sum_{z,w} G_N(0,z) G_n(z,w) =$ $4(\mathbb{E} I_N)^2$.

$(a+b)^2 \leq 2a^2 + 2b^2$

Lemma 2 pf. [we assume $a_N = \mathbb{E} I_N \to \infty$]

$\mathbb{E}_{x_0,y_0} I_N^2 = \sum_{z,w} \sum_{m,j=0}^{N} \mathbb{P}_{x_0}(X_m = z, x_j = w) \sum_{n,k=0}^{N} \mathbb{P}_{y_0}(y_n = z, y_k = w) \leq$ (same argument as Lemma 1)

$\sum_{z,w} [G_N(x_0,z) G_N(z,w) + G_N(x_0,w) G_N(w,z)][G_N(y_0,z) G_N(z,w) + G_N(y_0,w) G_N(w,z)] \leq$

$(a+b)(c+d) \leq 4(a^2+b^2+c^2+d^2)$

$4\sum_{z,w} G_N(x_0,z)^2 G_N(z,w)^2 + G_N(y_0,z)^2 G_N(z,w)^2 + G_N(x_0,w)^2 G_N(w,z)^2 + G_N(y_0,w)^2 G_N(w,z)^2 =$ $16 a_N^2$.

Also $\mathbb{E}_{x_0,y_0} I_N = \sum_{m,n=0}^{N} \sum_z P_{x_0}(x_m = z) P_{y_0}(y_n = z) = \sum_z G_N(x_0,z)\, G_N(y_0,z) \leq$ $\sqrt{\sum_z G(x_0,z)^2 \sum_z G(y_0,z)^2} = a_N$. Put $u_N(x,y) = \frac{\mathbb{E}_{x,y} I_N}{a_N} \leq 1$ $(\forall x,y)$, $u_N(x,x) = 1$.

$\forall x_0, y_0$ fix r s.t. $P^r(x_0,y_0) > 0$. $a_{r+N} = \sum_{m,n \leq N} \mathbb{P}_{0,0}(x_m = y_{r+n}) + \sum_{m \leq r+N} \sum_{n<r} \mathbb{P}_{0,0}(x_m = y_n) + \sum_{m<r} \sum_{n \leq N} \mathbb{P}(x_{N+m} = y_{r+n}) =$

$\mathbb{E}(a_N u_N(x_r, y_r)) \geq a_N[\mathbb{P}_r(x_0,y_0) u_N(x_0,y_0) + 1 - \mathbb{P}_r(x_0,y_0)]$; $o(a_N)$; $o(a_N)$

So we'll get our lemma.
Details: next time.