W6D4
3/12/14

Rand. Spanning Trees

Number of Trees in WSF/FSF.

observation for G recurrent, WSF = FSF is a single tree.

pf. Given an edge (x,y) $\mathbb{P}(x$ conn. to y in $B(x,k))$?
Consider G_n^*, run Wilson's alg. $x \to y$.
$$\limsup_n \mathbb{P}(\underbrace{\text{hit the } \cancel{\text{loop-erased SRW}}\ \partial B(x,k) \text{ before } y}_{A_k} \text{ in } G_n^W) = o_k(1).$$
$x \not\leftrightarrow y \Rightarrow$ No A_k holds, but $\mu^W(\neg A_k) \xrightarrow{k\to\infty} 0$,
so $\mu^W(x \leftrightarrow y) = 1$.

Wilson's algorithm rooted at infinity.

If $\{x_n\}$ is an infinite path in G s.t. any $v \in G$ occurs finitely many times, it's $LE(\{x_n\})$ is well-defined and is an ~~self-defined~~ infinite simple path.
$F_0 = \emptyset$, enumerate $V = \{x_1, x_2, \dots\}$ pick x_n, run an SRW until it hits F_{n-1} (if it does) or indefinitely and take $F_n = F_{n-1} \cup LE(\text{the path})$.

Prop The resulting forest's distributed μ^W, assuming G transitive.

Pf given a path $\{x_n\}$ s.t. $\forall v \in V$ occurs finitely many times, then $LE(\{x_n\}_1^L) \xrightarrow{L\to\infty} LE(\{x_n\}_1^\infty)$ in the sense that $\forall i \exists L(i) \forall L \geq L(i)$ $u_i^L = u_i$. Check the marginal of $e = (x,y)$.
Consider G_n^W run two RW in G $\{x_n\}, \{y_n\}$ starting at x, y. Let τ_x^n = hitting time of z_n, τ_y^n = hitting time of $LE(\{x_n\}_{n=1}^{\tau_x^n})$. $\tau_x^n \xrightarrow{n\to\infty} \infty$, $\tau_y^n \xrightarrow{n\to\infty}$ hitting time of y_n at $LE(\{x_n\})$. In G_n^W $\mathbb{P}(e \in UST) = \mathbb{P}(e \in LE(\{x_n\}_1^{\tau_x^n}) \cup e \in LE(\{y_n\}_1^{\tau_y^n})) \xrightarrow{n\nearrow\infty} \mathbb{P}(e \in LE(\{x_n\}_1^\infty) \cup LE(\{y_n\}_1^\infty)) \overset{\downarrow n \nearrow \infty}{=} \mu^W(e)$. The same

argument works for any finite set of edges, so all marginals are equal and the distribution is just μ^W.

<u>Cor</u> $\mathbb{P}(x, y \in$ SAME COMPONENT in WSF$) = \mathbb{P}($SRW from x intersect an indep. LERW from $y)$.

So, μ^W WSF is a tree if this event happens with probability 1. Fact (won't prove): If for some x, y, indep. SRW from x, y intersect with $\mathbb{P} = 1$, then WSF is connected.

<u>Thm</u> Let G be a transitive, transient graph. Write $G(x,y) = \mathbb{E}\left[\begin{smallmatrix}\#\text{ visits to } y \\ \text{of SRW from } x\end{smallmatrix}\right]$. If $\sum_{z \in V} G^2(\rho, z) = \infty$, then $\mathbb{P}(|\{x_m\} \cap \{y_n\}| = \infty) = \mathbb{P}(|LE(\{x_m\}) \cap \{y_n\}| = \infty) = 1$.

If the sum is $< \infty$ $\quad < 1$. We'll show

$\mathbb{E}(|\underbrace{\{x_n\} \cap \{y_n\}}_{\text{with multiplicity}}|) = \sum_{v \in V} \sum_{n,m} \mathbb{P}(x_m = z)\mathbb{P}(x_n = z)$