W6D3:
2/12/14

Random Spanning Trees

Infinite Electric Networks

$\ell^2_-(E)$, STAR $\perp$ CYCLE. $i^e_F = P_{CYCLE^\perp}\chi^e$, $i^e_w = P_{STAR}\chi^e$, and we saw a difference in the 3-reg. tree.

Prop 1 Let G be an infinite network with G_n exhausting sets, let $e \in E$, i_n current from e_- to e^+ in G_n. then $\|i_n - i^e_F\|_r \xrightarrow{n\to\infty} 0$, $\mathcal{E}(i^e_F) = r(e)\cdot i^e_F(e)$.

pf write $i_n = \chi^e - P_{CYCLE_n}\chi^e$. Enough to show $P_{CYCLE_n}\chi^e \xrightarrow{\ell^2} P_{CYCLE}\chi^e$, because $\text{CYCLE}_n \subset \text{CYCLE}_{n+1}$

this is clearly true. Note that the claim isn't true for STAR.

For the second part, $r(e)\, i^e_F = \langle i^e_F, \chi^e \rangle = \langle i^e_F, P_{CYCLE^\perp}\chi^e \rangle = \mathcal{E}(i^e_F)$.

Prop 2 Under the same conditions of Prop. 1, but i_n is in G^v_n, $\|i_n - i^e_W\|_r \xrightarrow{\ell^2} 0$, $\mathcal{E}(i^e_w) = r(e)\, i^e_w(e)$, and all other unit $e_- \to e^+$ currents have no larger energy.

Pf. Just the last part, as the rest is identical for Prop. 1. Any unit current θ is in CYCLE$^\perp$, and since STAR $\subset$ CYCLE$^\perp$, $\mathcal{E}(\theta) \geq \mathcal{E}(i^e_w)$.

Def. For any 2 vertices $u,v \in G$ the free/wired unit currents $i^{u\to v}_{W/F} = \sum_{i=1}^{\ell} i^{e_k}_{W/F}$ for $e_1, \dots e_\ell$ a $u \to v$ path.

Harmonic Dirichlet Functions

Recall for $f: V \to \mathbb{R}$ we've defined $df(e) = c_e(f(e^+) - f(e^-))$.
f is Dirichlet iff $df \in \ell^2_-(E)$.

Consider the Hilbert space of Dirichlet functions. Fix $\rho \in G$ and $\langle f, g \rangle = f(\rho)g(\rho) + \langle df, dg \rangle$ so that $f \equiv 0 \iff \langle f, f \rangle = 0$. The Dirichlet energy of f is just

$\langle df, df\rangle = \|df\|^2_r$.

Alternatively, just consider $D/\mathbb{R}$ with $\langle f+\mathbb{R}, g+\mathbb{R}\rangle = \langle df, dg\rangle$ and the Dirichlet energy of const functions f is $\|f\|^2_r = \langle f+\mathbb{R}, f+\mathbb{R}\rangle = \langle df, df\rangle = \|df\|^2_r$.

The ~~gradient~~ gradient is therefore an isometry $D/\mathbb{R} \to \mathrm{CYCLE}^\perp$.

If $\theta \in (\mathrm{STAR} \oplus \mathrm{CYCLE})^\perp$ then $d^{-1}\theta \in D/\mathbb{R}$ and is harmonic.
$\theta \in \mathrm{CYCLE}^\perp$ $\theta \in \mathrm{STAR}^\perp$

Denote the subspace of harmonic Dirichlet functions by HD.

So $\ell^2_-(E) = \mathrm{STAR} \oplus \mathrm{CYCLE} \oplus d(HD)$. Therefore, we get:

<u>Thm</u> $FSF = WSF \iff i_F^e = i_W^e \ \forall e$ (unique currents) $\iff \ell^2_-(E) = \mathrm{STAR} \oplus \mathrm{CYCLE} \iff$

$\iff HD = \{\text{const. functions}\}$.

<u>We proved</u> that any tran. am. graph has no non-const HD.

<u>A criterion for uniqueness of currents:</u>

<u>Def.</u> Given a finite subnetwork A of G,

$$RD(A) = \sup\{R_{eff}(x \leftrightarrow y; A) \mid x, y \in A\}.$$

<u>Thm</u> If $\{W_n\}$ are finite, pairwise edge disjoint subnetworks of G, each separating each vertex from infinity (any simple path $x \to \infty$ intersects all but finitely many W_ns). If $\sum \frac{1}{RD(W_n)} = \infty$ then the currents are unique.

<u>pf.</u> Let f be harmonic & non-constant. We'll show $\|df\| = \infty$. Let e_0 be an edge on which $df(e_0) \neq 0$, and let $n_0 \in \mathbb{N}$ s.t. all W_n, $n > n_0$ separate e_0 from ∞. Put $G_n = W_n +$ all vertices it separates (finite subgraph). Let x_n, y_n be the pts. in G_n where f is max. and min. By the max./min. principal this is attained on W_n.

Put $F_n = \frac{f - f(y_n)}{f(x_n) - f(y_n)}$, then $|dF_n| \leq \frac{|df|}{f(x_n) - f(y_n)} \leq \frac{|df|}{|df(e_0)|}$

But $\frac{1}{RD(W_n)} \leq \frac{1}{R_{eff}(x_n, y_n) \text{ in } W_n} \leq \sum_{e \in W_n} c_e \frac{df(e)^2}{df(e_0)^2}$, so

$$\mathcal{E}(f) \geq \sum_{n=1}^{\infty} \sum_{e \in W_n} c_e\, df(e) \geq |df(e_0)|^2 \cdot \sum_{n=1}^{\infty} \frac{1}{RD(W_n)} = \infty, \quad \text{So } \|df\| = \infty.$$

<u>Cor.</u> $\mathbb{Z}^d$ has unique currents [or any other "weak" product of infinite connected graphs].

<u>pf.</u> Just take $W_n = \boxed{\square_n}$. $RD(W_n) \leq 4n$.