V5 D4   Random Spanning Trees

26/11/14

G is edge-ameanable if $\exists$ finite V_n s.t. $\inf_n \frac{|\partial_E V_n|}{|V_n|} = 0$. Can always take V_n to be exhausting. $\forall v,u \exists$ Graph automorphism which moves v to u

__Thm.__ G transitive & edge-ameanable then $\mathbb{E}\deg_F(x) = 2$ where F is the FSF/WSF.

[In fact, for WSF $\mathbb{E}\deg_F(x) = 2 \Longleftrightarrow$ F-a.s. the component of x is recurrent.]

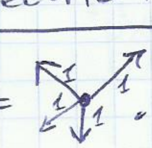
When G is weighted we replace $\frac{|\partial_E V_n|}{|V_n|}$ by $\frac{\sum_{e \in \partial_E V_n} c_e}{\sum_{x \in V_n} \pi_x}$.

__pf.__ Let F be a forest in G in which all trees are ∞. Let V_n be sets as in edge-ameanability def., with G_n spanned by V_n. Let $k_n = \#$trees in $F \cap G_n$. Any such tree must intersect $\partial_E V_n$, so by edge-ameanability $\frac{k_n}{V_n}$. Let $W = F \cap G_n$. Then $\underbrace{\sum_{x \in V_n} \deg_W(x)}_{2(|V_n| - k_n)} \le \sum_{x \in V_n} \deg_F(x) \le \underbrace{\sum_{x \in V_n} \deg_W(x)}_{2(|V_n| - k_n)} + |\partial_E V_n|$ so after dividing by $|V_n|$ we get $\lim_{n \to \infty} \frac{1}{|V_n|} \sum_{x \in V_n} \deg_F(x) = 2$.

So, for a random forest F in a bounded degree graph, by DCT $\lim_{n \to \infty} \frac{1}{|V_n|} \sum_{x \in V_n} \mathbb{E}\deg_F(x) = 2$. F is invariant to automorphisms and G transitive, $\mathbb{E}\deg_F x = 2$.

__cor__ on a trans. ameanable graph (in particular $\mathbb{Z}^d$) WSF=FSF by coupling.

__Infinite Electric Network - Linear Algebra__

Star(v) = 

. STAR = $\overline{\text{Span}(\text{Star}(v))}$. Similarly CYCLE...

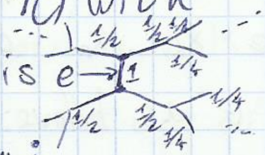
We work on $\ell^2_-(E,r) = \{$antisymm. funcs. θ on E with $\sum_{e\in E} \theta(e)^2 r_e < \infty\}$.

We still have STAR $\perp$ CYCLE, but not necessarily STAR $\oplus$ CYCLE $= \ell^2_-(E,r)$. This leads to 2 natural defs:

$i_F^e = P_{CYCLE^\perp} \chi^e$, $i_W^e = P_{STAR} \chi^e$.

Instructive example: compute i_F^e, i_W^e on the 3-reg. tree.

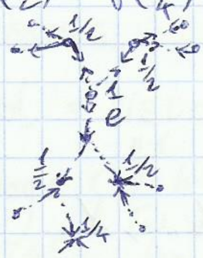
CYCLE is trivial so $i_F^e = \chi^e$. To compute i_W^e we'll write $\chi^e = \frac{1}{3}(\theta + \eta)$ with $\theta \perp$ STAR, $\eta \in$ STAR. θ is



by symm., with energy:

$1 + 4\left(\frac{1}{2}\right)^2 + 8\left(\frac{1}{4}\right)^2 + \dots < \infty$.

η:



and it's easy to see it is also of finite energy.

and $\frac{1}{3}(\theta + \eta) = \chi^e$, so $i_W^e = \frac{2}{3}$.