28/10/14
W1D3

Uniform Spanning Trees

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Teu. $14^{00}-16^{00}$

Wed. $14^{00}-15^{00}$

Grading - HW assignments (3-5 sheets).

Book: Probability on Trees and Networks (Lyons & Peres)

Def: A tree is a connected graph with no cycles.

Def: Given a graph $G=(V,E)$ a spanning tree of G is a tree on V with edges from E.

If G is connected there's at least 1 sp. tr. - we can draw one at random, uniformally.

We'll try to create "Infine sp. tr." on $\mathbb{Z}^2$...

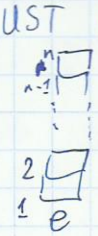
Example: $G = K_n$. thm (Cayley) - G has n^{n-2} sp. tr.

Typical question - what is the typical distance between v_1 and v_2 in the UST of K_n? turns out it's about $\sqrt{n}$.

Thm. (Kirchoff) gives a formula for $\#\{\text{UST of } G\}$ for any finite G. We'll prove that later in the course.

Ques. Given a graph G and $e \in E(G)$, how to comp. $\mathbb{P}(e \text{ is in UST})$?

$\#(\text{UST of } G \text{ after contracting } e\} / \#(\text{UST of } G)$

Example: $G_n =$  . $\mathbb{P}(e \in \text{UST}(G_n)) = a_n$

$a_1 = 1, a_2 = 3/4, a_3 = ?$...

In fact, $a_n \to \sqrt{3} - 1 \approx 0.73$

Electric Network intuition -  each edge has $\rho = 1$ Ohm.

The current on e = this $\mathbb{P}$, as it turns out. (so $e_n \searrow \sqrt{3}-1$).

We'll use many connections to random walk theory.

How to define a UST of $\mathbb{Z}^2$? Using a measure limit of USTs on finite subgraphs whose union is G (should be monotone: $G_{n-1} \subseteq G_n$). What is that limit? If $e_1 \dots e_n \in \mathbb{Z}^d$ -

$\mu_n(e_1 \in UST, \dots e_n \in UST)$ convergence. To what? Some prob. measure μ

by colmogorov's consistency theorem. μ is concentrated on graphs with no cycles. Is it concentrated on spanning trees?

<u>Thm.</u> Yes, iff $d \le 4$. What about other graphs (infinite)?

μ is called the <u>free UST</u>. (FUSF)

There are other ways to define it - replace the infinite component with a single vertex, for example. 

μ^w - <u>wired UST</u> (WUSF). Also concentrated on spanning forests.

<u>Question</u> Is $\mu^w = \mu$? Yes for $\mathbb{Z}$, no for the 3-regular infinite tree.

In fact, yes for $\mathbb{Z}^d$. In some sense, μ is larger.

We can couple them so that WUSF $\subseteq$ FUSF.

Now, Back to the finite world - how do we <u>algorithmically sample</u> a UST? What about extending it to the infinite case? Many connections to random walks.

<u>Aldous-Broder:</u> Start a simple random walk in G add every edge you traverse unless it closes a cycle in the subgraph.

There are, in fact, a faster algorithm.

<u>Wilson's Algorithm</u>

<u>Def:</u> P a finite path in G, $x_0 \dots x_\ell$. Its Loop-Ereasuere $LE(P) = u_0, \dots u_m$ is obtainet by easing cycles in the order they're created. Formally - $u_0 = x_0$. If $x_\ell = x_0$ put $m=0$ and terminate. Otherwise, let u_1 be the first vertex of x after the last visit to x_0. $u_1 = x_{i+1}$ where $i = \max_{\substack{x_j = x_0 \\ 0 \le j \le \ell}} j$. If $x_\ell = u_1$ put $m=1$ and terminate, ...

<u>Wilson's Alg.</u> Order $V = \{v_1, \dots v_n\}$ and start creating a seq. of trees. $T(0) = \{v_1\}$. If T_i spans G stop, otherwise take LE(P) of random walk P from $\min_{v_j \notin T_i} j$ to T_i.

Cor. UST path between x, y is destributed as the LER of a simple randomwalk from x to y.