Graph Theory: Homework Assignment Number 3

Due: Wednesday, January 21, 2009.

1. Let G be a simple Hamiltonian graph on n vertices in which every degree is at least $k+2$, and suppose the independence number of G is at most k. Prove that G contains a cycle of length precisely $n-1$.

   Hint: Take a shortest chord in a Hamilton cycle; if it does not provide a cycle of length $n-1$ apply the argument in the proof of the Chvatál-Erdös Theorem.

2. Let $V = A_1 \cup A_2 \cup \cdots \cup A_k$ be a partition of a finite set V of sk elements into k pairwise disjoint sets A_i, each of size s. Let $V = B_1 \cup B_2 \cup \cdots \cup B_k$ be a similar partition, where each B_j is also of size s. Prove that there is a set $X \subset V$, $|X| = k$ containing exactly one element from each set A_i and from each set B_j.

3. Prove that the edges of every bipartite graph with minimum degree δ can be colored by δ colors so that every vertex is incident with an edge of every color.

4. (i) Is there a finite n so that every simple, connected graph on at least n vertices contain an **induced** subgraph with precisely 15 edges ? Prove or supply a counter-example.

   (ii) Is there a finite n so that every simple, connected graph on at least n vertices contain an **induced** subgraph with precisely 17 edges ? Prove or supply a counter-example.

5. Let A be a set of $3m$ points in the Euclidean plane, and suppose that the distance between any two of these points is smaller than $\sqrt{2}$. Prove that the number of pairs P, Q of points of of A so that the distance between P and Q is at least 1 does not exceed $3m^2$.

6. Let G be a simple graph with maximum degree 7 containing no clique of size 4. Prove that the chromatic number of G is at most 6.

   Hint: Show first that one can delete from G a bipartite graph leaving each degree in what's left at most 3.