Graph Theory: Homework Assignment Number 1

Due: Wednesday Dec. 10.

1. Prove that every simple graph with $n \geq 2$ vertices contains two vertices of the same degree.

2. Show that every simple graph with $n \geq 7$ vertices and at least $5n - 14$ edges contains a subgraph with minimum degree 6.

3. Show that in a tree T on at least two vertices, the number of leaves exceeds the number of vertices of degree at least three by at least 2.

4. What is the number of spanning trees in the complete bipartite graph $K_{50,50}$?

5. Let $\pi = (\pi_1, \pi_2, \ldots, \pi_{100})$ and $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_{100})$ be two permutations of the numbers $1, 2, \ldots, 100$. Show that there is a function $f : \{1, 2, , \ldots, 100\} \mapsto \{-1, 1\}$ such that for every $1 \leq i < j \leq 100$ the following two inequalities hold:

$$|\sum_{r=i}^{j} f(\pi_r)| \leq 2 \quad \text{and} \quad |\sum_{r=i}^{j} f(\sigma_r)| \leq 2.$$

   Hint: Consider the graph on the set of vertices $\{1, 2, \ldots, 100\}$, whose set of edges is

$$\{ \{\pi_{2i-1}, \pi_{2i}\} : 1 \leq i \leq 50\} \cup \{ \{\sigma_{2j-1}, \sigma_{2j}\} : 1 \leq j \leq 50\}.$$