WGD2
1/12/14

Geometry of Numbers

Space of lattices

Goal: Put a topology on $\mathcal{L}_n = SL_n(\mathbb{R})/SL_n(\mathbb{Z})$ $(gSL_n(\mathbb{Z}) \to g\mathbb{Z}^n)$

" " " " $\{\Lambda \subset \mathbb{R}^n \mid \Lambda \text{ a lattice}\} = GL_n(\mathbb{R})/GL_n(\mathbb{Z})$

Integer matrices of det. ±1.

Those are homogeneus spaces. Can define the quotient topology inherited from a topology on $SL_n(\mathbb{R})$ as a subspace of $\mathbb{R}^{n^2}$. This won't give us much info about it as a top. space.

Some equivalent definitions - (since $[SL_n(\mathbb{R})]$ (GL) is a Lie group and $[SL_n(\mathbb{Z})]$ (GL) is discreet)

Let $v_1, \dots, v_n$ be lin. ind., $\Lambda = \mathbb{Z}v_1 + \dots + \mathbb{Z}v_n$ [with $d(\Lambda)=1$] and let $\varepsilon > 0$, $U(v_1, \dots, v_n, \varepsilon) = \{\mathbb{Z}u_1 + \dots + \mathbb{Z}u_n \mid u_1, \dots u_n \text{ lin ind } \forall i \ \|v_i - u_i\| < \varepsilon\}$ [with $d(\Lambda')=1$] is a basis for the top.

Equivalence - exercise.

We can define a notion of convergence $\Lambda_n \to \Lambda$ using this topology.

Prop. $\Lambda_j \to \Lambda \Rightarrow$ For any finite $F \subseteq \Lambda$ and $\varepsilon > 0$ $\exists j_0 \ \forall j > j_0$, $v_j \in \Lambda_j$ with $\|f - v_j\| < \varepsilon$. In particular, $\forall v \in \Lambda, \exists v_j \in \Lambda_j$ s.t. $v_j \to v$.

pf choose bases $v_1, \dots v_n$ of Λ, $v_1^{(j)}, \dots, v_n^{(j)}$ of Λ_j s.t. $v_i^{(j)} \xrightarrow[j\to\infty]{} v_i$. For each $f \in F$ write $f = \sum a_i v_i$, so $f_j = \sum a_i v_i^{(j)} \xrightarrow[j\to\infty]{} f$, so $\exists j_0(f)$ s.t. $\forall j \geq j_0(f)$, $\|f_j - f\| < \varepsilon$.

Now, simply take $j_0 = \max_{f \in F} j_0(f)$.

Prop If $\Lambda_j \to \Lambda$ then $\Lambda = \{\lim_{j\to\infty} v_j \mid v_j \in \Lambda_j \text{ and the limit exists}\}$.

Pf. Denote the set of limits Λ'. We've just seen $\Lambda \subseteq \Lambda'$.

For $\supseteq$, suppose $x \in \mathbb{R}^n$, $\exists u_j \in \Lambda_j$ s.t. $u_j \xrightarrow[j\to\infty]{} x$. Choose bases $v_1, \dots, v_n$ for Λ, $v_1^{(j)}, \dots, v_n^{(j)}$ for Λ_j s.t. $v_i^{(j)} \xrightarrow[j\to\infty]{} v_i$. Let A_j be the lin. trans. $\forall i, v_i^{(j)} \to v_i$, then $A_j \xrightarrow[j\to\infty]{} I$. Let $y_j = A_1^{-1}(A_j u_j) \in \Lambda_1$, but $A_j u_j \xrightarrow[j\to\infty]{} x$, so $y_j \xrightarrow[j\to\infty]{} A_1^{-1}x$. In particular y_j's are constants from some point on - say $\forall j > k \ y_j = y_k$. Write $y_j = y_k = \sum a_i v_i^{(1)}$, so

$u_j = A_j^{-1} A_1 y_j = \sum a_i v_i^{(j)}$. So $x = \lim_{j\to\infty} u_j = \sum a_i \lim_{j\to\infty} v_i^{(j)} = \sum a_i v_i \in \Lambda$.

What about such limit sets where Λ_j doesn't converge?

Example - $\Lambda_j = \mathbb{Z}(2^j, 0) + \mathbb{Z}(0, \frac{1}{2^j}) \in \mathcal{L}_2$. The set of limits is $\{0\}\times\mathbb{R}$.

Chabauty Topology (AKA Gromov-Hausdorf topology on subgroups of $\mathbb{R}^n$). Let $\mathrm{Sub}(\mathbb{R}^n)$ be all subgroups of $\mathbb{R}^n$.

Define $d(\Lambda, \Lambda') = \inf\{\rho : B(0, 1/\rho) \cap \Lambda' \subseteq \Lambda^{(\rho)}\}$. Thm. This is a matric, is G is locally compact, $\Lambda^{(\rho)} = \bigcup_{\lambda\in\Lambda} B(\lambda, \rho)$ $\mathrm{Sub}(G)$ is a compact metic space. Our topology is, in fact, a restriction of the chabauty Topology to the space of lattices (in particualar, matrizable).

Def If X top. space, $X_0 \subseteq X$ is bounded if $\overline{X_0}$ is compact.

Thm. (Mahler compactness criterion) $X_0 \subseteq \mathcal{L}_n$ is bounded $\iff$

$\exists r>0$ s.t. $\forall \Lambda\in X_0$, $B(0,r)\cap\Lambda = \{0\} \iff \forall \Lambda \in X_0, \lambda_1(\Lambda) \geq r$.

Equiv., $\{\Lambda_j\}_{j\in\mathbb{N}} \subseteq \mathcal{L}_n$ has no convergent subsequence $\iff$ $\lambda_1(\Lambda_j) \xrightarrow[j\to\infty]{} 0$.

Pf. $\Leftarrow$ suppose $\lambda_1(\Lambda_j) \xrightarrow[j\to\infty]{} 0$ and Λ_j convergent, $\Lambda_j \to \Lambda$. Let $r = \lambda_1(\Lambda) > 0$. Since $\lambda_1(\Lambda_j) \to 0$, $\exists v_j \in \Lambda_j$ for all sufficiently large j, with $\|v_j\| < \frac{r}{3}$, so $\exists u_j \in \Lambda_j$ with $\frac{r}{3} \leq \|u_j\| \leq \frac{2r}{3}$. So u_j has a convergent subsequence with $\lim_{i\to\infty} u_{j_i} = u \in \Lambda$, but $\|u\| \leq \frac{2r}{3} < r = \lambda_1(\Lambda)$. #

$\Rightarrow$ Recall Minkowski's 2nd thm - $\lambda_1 \cdots \lambda_n \leq C$ (where C is some const. dependent on n), $\mu_1 \cdots \mu_n \leq C'$.

Λ contains a basis of vectors of length at most μ_n.

Assume by contradiction $\lambda_1(\Lambda_j) \geq r > 0$. Chooses bases $v_i^{(j)}$ with $\|v_n^{(j)}\| \leq \mu_n(\Lambda_j) \leq \frac{C'}{\mu_1\cdots\mu_{n-1}} \leq \frac{C'}{\lambda_1(\Lambda_j)^{n-1}} \leq \frac{C'}{r^{n-1}}$. So, in some subsequence, $v_1^{(j)} \xrightarrow[j\to\infty]{} v_1$, $v_2^{(j)} \xrightarrow[j\to\infty]{} v_2, \ldots$ Let Λ be ~~the lattice~~ spanned by $v_1, \ldots, v_n$. But $\det\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} = \lim_{j\to\infty} \det\begin{pmatrix} | & & | \\ v_1^{(j)} & \cdots & v_n^{(j)} \\ | & & | \end{pmatrix} = \lim_{j\to\infty} 1 = 1$ so $\Lambda \in \mathcal{L}_n$

Thm. (Mahler non-unimodular) $X_0 \subseteq \{\text{Space of lattices}\}$ is bounded iff (ii) $\sup_{\Lambda \in X_0} d(\Lambda) < \infty$, (i) $\inf_{\Lambda \in X_0} \lambda_1(\Lambda) > 0$.

pf. rescaling of prev. thm.

$\Delta_n = \inf(d(\Lambda) \mid \Lambda \cap B(0,1) = \{0\})$. The following will show the inf is actually a min.

prop. Let A be a bounded subset or $\mathbb{R}^n$ containing a ball around the origin. $\Delta(A) = \inf\{d(\Lambda) ; \Lambda \cap A^\circ = \{0\}\}$. Then $\exists$ lattice Λ with $\Delta(A) = d(\Lambda)$, $\Lambda \cap A^\circ = \{0\}$.

pf. Let $\Lambda_j < \mathbb{R}^n$ be lattices with $d(\Lambda_j) \searrow \Delta(A)$. By Mahler, Λ_j has a converging subsequence, say $\Lambda_j \xrightarrow[j\to\infty]{} \Lambda_0$. If $v \in A^\circ \cap \Lambda_0$, $v \neq 0$, by def. $v \in \Lambda_j$ for sufficiently large j.

We used that when calculating Δ_2:

1. Assumed inf is achived by Λ.
2. Showed it had 3 pairs of points with $\{x^2+y^2=1\}$ (pertubation).
3. Showed Λ is hex. lattice up to rotations.

Equiv. to finding Δ_n: $\overset{\sup}{\max}\{\lambda_1(\Lambda) \mid \Lambda \in \mathcal{L}_n\} = \lambda_1^{(n)} = \Delta_n^{-1/n}$.

let Q be a pos. def quad. form, $\gamma(Q) = \inf_{\substack{x \neq 0 \\ x \in \mathbb{Z}^n}} \frac{Q(x)}{\sqrt[n]{\det(a_{ij})}}$.

Define $\gamma_n = \overset{\max}{\sup}\{\gamma(Q) \mid Q \text{ pos. def.}\}$. $\gamma_n = (\lambda_1^{(n)})^2$. γ_n is the Hermite const.

$\gamma_1, \dots, \gamma_8$ are known and so is γ_{24}. They are achieved on unique (up to scaling) quad. form.

Minkowski I $\to \Delta_n \geq \frac{V_n}{2^n}$, so $\gamma_n \leq C \cdot n$. We'll see later $cn \leq \gamma_n$. It isn't known weather $\lim \frac{\gamma_n}{n}$ exists.

Best known bounds - $\frac{1}{2\pi e} \leq \liminf \frac{\gamma_n}{n}$, $\limsup \frac{\gamma_n}{n} \leq \frac{1}{\pi e}$. Also open: If γ_n increasing?

Def. Q $\overset{\text{pos. def}}{\text{quad. form}}$ is <u>extreme</u> if $\forall Q_1 \subseteq \{\text{Some neighbourhood of } Q\}$. Q_1 is either tQ, or $\gamma(Q_1) < \gamma(Q)$. Let Sym_n be the space of symm. matrices, symm_n^+ (pos. def. sym).

remark - (a_{ij}) pos. def $\iff \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix} > 0$ $\vdots$ $\det(a_{ij}) > 0$

def. Q pos. def. quad. form, $\lambda = \min_{x \in \mathbb{Z}^n \setminus \{0\}} Q(x)$

with min. attained at $\pm u_1, u_2, \dots u_s$. Q is perfect if it's uniquely determined by $\lambda = Q(u_l)$.

Note: suppose $Q_1(x) = \sum b_{ij} x_i x_j$ s/t $Q(u_l) = Q(u_l)$, so
$0 = Q_1(u_l) - Q_2(u_l) = \sum (b_{ij} - a_{ij}) u_i^{(l)} u_j^{(l)}$. So Q perfect $\iff$
no solutions $t_{ij} = t_{ji}$ to $\sum t_{ij} u_i^{(l)} u_j^{(l)} = 0$.

Given Q as above let $\tilde{A} = (\tilde{a}_{ij}) = \operatorname{adj} A$. $Q(x) = \langle \tilde{A}x, x\rangle$

Def. Q is eutactic if $\exists \rho_1, \dots \rho_s > 0$ s.t. $\tilde{Q}(x) = \sum_{l=1}^{s} \rho_l \langle u_l, x\rangle^2$
($\pm u_1, \dots \pm u_s$ all as before).

THM (Voronoi) Q extreme $\iff$ Q perfect & eutactic.

We'll prove that next week.