W3 D2

10/11/14

Geo. of Nums.

Reminder - Minkowski's thm. $A\subseteq\mathbb{R}^n$ conv. and cent.-symm. with $Vol(A)\geq 2^n d(\Lambda)$

then $S\cap(\Lambda\setminus\{0\})\neq\emptyset$.

Cor. (Minkowski's lin. forms thm) - $L_i:\mathbb{R}^n\to\mathbb{R}$ linear, $i=1,\dots,n$, $L_i(x)=\sum_{j=1}^{n}a_{ij}x_j$. let $D=|\det a_{ij}|$. For $c_1,\dots c_n$ s.t. $\prod_{i=1}^{n}c_i<D$

then $\exists\,\vec{x}\in\mathbb{Z}^n\setminus\{0\}$ s.t. $|L_i(\vec{x})|<c_i$, $i=1,\dots n$.

pf. the vol. of $\{\vec{x}\mid\forall i, |y_i|<c_i\}$ is $2^n\prod_{i=1}^{n}c_i$, and $L(\mathbb{Z}^n)$ is a lattice of co-volume $2^n D$.

Let P_n = packing radius in $\mathbb{R}^n$ = $\sup\{t>0\mid\exists\,\Lambda\in\mathcal{L}_n$ s.t. balls of radius t centered at Λ are disjoint$\}$

{all n-dimensional lattices of $d=1$}

P_n is known for $n=1,\dots 8$ and $n=24$, But the proofs are complicated.

Kepler's conj.  pyramidal packings are optimal. (of all packings, not just lattices)

difficult, computer-assisted proof was reently found.

Thm Let V_n be the vol. of the n-dim. unit ball. Then, $P_n\leq V_n^{-1/n}$.

pf. suppose $P_n> V_n^{-1/n}$. So $\exists\,\Lambda\in\mathcal{L}_n$ (with $d(\Lambda)=1$) s.t. balls of radius $r>V_n^{-1/n}$ have disjoint translates, or $B(0,2r)\cap\Lambda=\{0\}$. This has vol. $2^n r^n V_n>2^n$, so we get a contradiction.

Fact $V_n=\frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$, $\frac{1}{\sqrt[n]{V_n}}\sim\frac{\sqrt{n}}{\sqrt{2\pi e}}$.

Let $S\subseteq\mathbb{R}^n$. If $S\cap(\Lambda\setminus\{0\})=\emptyset$ then S is admissble for Λ (or Λ for S). The lattice const $\Delta(S)=\inf_{\substack{\Lambda\in\mathcal{L}_n\\ \Lambda\text{ is admis.}}}\{d(\Lambda)\}=[\sup\{t>0\mid\exists\,\Lambda\in\mathcal{L}_n$ s.t. tS is admissible$\}^{-n}]$. If $S=B(0,1)$ the euc. norm n-dim. unit ball, $\Delta_n=\Delta(S)$. From Minkowski, $\Delta_n\leq 2^n V_n$.

Let's compute Δ_2.

Thm. $\Delta_2=\frac{\sqrt{3}}{2}$, inf in def. is actually min, realized by $\Lambda=\mathbb{Z}(1,0)\oplus\mathbb{Z}(\cos 60°, \sin 60°)$ scale appropriately.

The hexagonal lattice. Up to rotations the minimizer is unique.

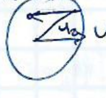
Remark: At this point we assume the inf is a min. Proof- later, as part of a more general theory ($\forall n$, $\forall S$ convex & contains a neighbourhood of 0). Let Λ_c s.t. $d(\Lambda_c)$ is the inf.

Lemma: Λ_c contains 3 distinct pairs $v_1,-v_1,\ldots$ v's in $B(0,1)$

Pf. If $\Lambda_c\cap B(0,1)=\{0\}$ and $\Lambda_c\cap S^1=\emptyset$ we could expand the ball/shrink Λ and get lower inf.

If $\Lambda_c\cap S^1=\{v_1,-v_1\}$ we can assume Λ_c has $(\pm 1,0)$. Replace Λ_c by $\begin{pmatrix}1&0\\0&t\end{pmatrix}\Lambda_c$ $t<1$, cont.

If $\Lambda_c\cap S^1=\{\pm u_{1,2}\}$. Let $u_1^\pm=\pm(1,0)$, $u_2^\pm=\pm(\cos\theta,\sin\theta)$. $\theta\geq\frac{\pi}{3}$.

O/W  u_2-u_1 would be in $B(0,1)$.

[Lemma u_1,u_2 are a basis of Λ_c] $d(\Lambda)=\sin\theta$ from lemma. If $\theta>\frac{\pi}{3}$ a new lattice $u_1,(\cos\theta',\sin\theta')$, $\frac{\pi}{3}<\theta'<\pi$. For $\theta'-\theta$ small the new lat. has cov. $\sin\theta'<\sin\theta$. Critical $\Leftrightarrow$ $u_1, u_2=R_{0,\pi/3}(u_1)$ so it also has $-u_1+u_2=(\cos\frac{2\pi}{3},\sin\frac{2\pi}{3})$

So $\Lambda_c\cap S^1$ has at least 6 vectors. Any angle between 2 has $\theta\geq\frac{\pi}{3}$ so there are exactly 6, $\Lambda_c=$ hex. lat. By comp. $d(\Lambda_c)=\sin\frac{\theta}{3}=\frac{\sqrt{3}}{2}$

Still have to prove sublemma: equiv. u_1,u_2 are the shortest vectors in Λ - but this isn't enough to generate. In $\mathbb{R}^5$, Λ lattice $\langle e_1,\ldots,e_5,\underbrace{(\tfrac12,\ldots,\tfrac12)}_{u}\rangle$. $\|u\|=\sqrt{\frac{5}{4}}>1$ but $e_1,\ldots e_5$ shortest won't generate Λ. Back to sublemma $\pm(u_1,0)=\pm(1,0)$ project onto y axis. Get discrete subgroup of $\mathbb{R}$. Take y_0 it's smalest. We claim $y_0\geq\sin\frac{\theta}{3}=\frac{\sqrt{3}}{2}$.

If it's not a base we have $0<y_0<\frac{\sqrt{3}}{2}$, so $y_2+nu_1\in\Lambda$ has length <1. Let $u_2=(x_2,y_0)$. $ny_0=y_2$, but then $y_2<1$ so $y_0=y_n$.

$y_2=|\det(u_1,u_2)|=y_0=d(\Lambda)$.

Some Generalities in Lattices

Suppose Λ, M are lattices $\Lambda \subset M$. sublattice

Prop take a $v_1, \dots v_n$ basis of M. then $\exists u_1, \dots u_n$ basis of Λ of the form (*) $u_1 = a_1 v_1$, $u_2 = a_{21} v_1 + a_{22} v_2, \dots$

$a_{ij} \in \mathbb{Z}, a_{ii} \neq 0$.

Prop Any basis u of Λ has some basis v of M that satisfies (*).

Pf $D = [M : \Lambda]$. Then $D v_i \in \Lambda$. Λ has elem. of form $a_1 v_1 + \dots + a_{ii} v_i$, $a_{ii} \neq 0$. Choose u_i a vector of this form will $|a_{ii}| > 0$ as small as possible. Want to show $u_1, \dots, u_n$ genrate Λ. o/w $w \in \Lambda \setminus \{\mathbb{Z} u_1 \oplus \dots \oplus \mathbb{Z} u_n\}$. Write $w = t_1 v_1 + \dots t_k v_k$, $t_k \neq 0$ (since $w \in M$). Choose w s.t. k is lowest possible, $|t_k|$ " " ". We can substract u_k until $t_k = 0$ (o/w well get a smaller a_{kk}) and go on until we get $w = q_1 u_1 + \dots + q_k u_k$. The opposite direction is similar.

Cor. 1 Can take $a_{ii} > 0$, $0 \leq a_{ij} < a_{jj}$ (or a_{ii}). coeffs of u.

pf $u_i' = \sum c_{i,j} u_j$. $d_{i,j} = c_{i,j} a_{jj} + c_{i,j+1} a_{j+1 j} + \dots + a_{ij}$.

Choose c_{ij} sequentially by long division in a_{jj}.

Cor. 2 $u_1, \dots u_m$ lin. ind. $\to \exists v$ basis of M s.t. (*) holds, $0 \leq a_{ij} \leq a_{ii}$.

pf Cor 1 for $u_1, \dots u_m, \underbrace{u_{m+1}, \dots u_n}_{\text{basis completion.}}$ and apply cor. 1.

cor. 3 TFAE: $u_1, \dots u_m$ lin. in. in M -

1) they can be completed to a basis of M.
2) $\mathrm{Span}_{\mathbb{R}}(u_i) \cap M = \mathbb{Z} u_1 \oplus \dots \oplus \mathbb{Z} u_m$.

Def $u_1, \dots u_m$ that satisfy that are a primitive set.

pf $1 \Rightarrow 2$ $\supset$ immediate. $c \in \mathrm{Span}_{\mathbb{R}}(u_i)$, $c = \sum_{i=1}^{m} \overset{\in \mathbb{R}}{a_i} u_i = \sum_{i=1}^{n} \overset{\in \mathbb{Z}}{k_i} u_i$ so by lin. ind. $c = \sum_{i=1}^{m} k_i u_i$.

$2 \Rightarrow 1$ Given $u_1, \dots u_m$ apply cor 2 to find $v_1, \dots v_n$ and let a_{ij} be the cofficients. $v_i \in \mathrm{span}(u_1, \dots u_i) \cap M$ so it's an integer comb. of $u_1, \dots u_m \Rightarrow a_{ii} = 1$. So $v_j = u_j$ for $1 \leq j \leq m$ and therefore v is a completion of u to a basis.

What's Nexts? Connecting Num. Fields & lattices

Num. field $\rightarrow$ lat.

Let k be a totally real num. field of deg n, $\sigma_1, \dots \sigma_n$ distinct real embeddings. $\mathcal{O}_k$ ring of integers in k, $\mathbb{Z}\alpha_1 \oplus \dots \mathbb{Z}_n \alpha_n$. The geom. embedding of $\mathcal{O}_k$ is a lattice with basis $(\sigma_1(\alpha_j), \dots \sigma_n(\alpha_j))_{j=1}^{n} \subseteq \mathbb{R}^n$.

Def. $k \subseteq \mathbb{C}$ is a num. field if $\deg(k/\mathbb{Q}) = n < \infty$.

Fact any element x of a num. field is algebraic $- \exists P \in \mathbb{Z}[x]$, $P(x) = 0$, since $1, x, x^2, \dots x^n$ are lin. dep.

Def Algebraic integer is a root of $P \in \mathbb{Z}[X]$ with leading coeff. 1. $\sqrt{2}$ is, $\sqrt{\frac{2}{3}}$ isn't. The set of algebraic integers is a ring - the integers in k are a ring in k called the integer ring in k, $\mathcal{O}_k = \mathbb{Z}\alpha_1 \oplus \dots \oplus \mathbb{Z}\alpha_n$ n-generated additively. $n = \deg(k/\mathbb{Q})$

Fact $\deg(k/\mathbb{Q}) = \#\{\sigma : k \xrightarrow[\text{field emb.}]{} \mathbb{C}\}$. k is totally real if any σ maps k to $\mathbb{R}$