W13D2

19/1/15

Geom. of Numbers

$N:\mathbb{R}^n \to \mathbb{R}$, $N(x_1,\dots,x_n) = |\prod_{i=1}^n x_i|$. $A = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \in SL_n(\mathbb{R}) \right\}$, N is A-invariant.

<u>Minkowski's conjecture</u> (I) $\forall \Lambda \in \mathcal{L}_n, x \in \mathbb{R}^n \ \exists y \in \Lambda$ s.t. $N(x-y) \leq 1/2^n$.

(II) If $\Lambda \notin A\mathbb{Z}^n$ $\mu(\Lambda) = \sup_{x\in\mathbb{R}^n} \inf_{y\in\Lambda} N(x-y) < 1/2^n$.

(I) + (II) are equivalent to (I') + (II), where

(I') $\forall \Lambda \in \mathcal{L}_n, \mu(\Lambda) \leq 1/2^n$.

Proven in dimension ≤ 9 using the strategy from last week.

<u>prop.</u> ⊛ $a \in A$, $\mu(\Lambda) = \mu(a\Lambda)$.

⊛ $\Lambda_n \to \Lambda$ then $\limsup \mu(\Lambda_k) \leq \mu(\Lambda)$

⊛ $\Lambda' \in \overline{A\Lambda} \Rightarrow \mu(\Lambda') \leq \mu(\Lambda)$.

<u>pf</u> 1 follows from A invariance, 3 from 1+2. 2 left as an exercise, follows directly from the definition of topology on $\mathcal{L}_n$.

<u>prop.</u> (Remak - Davenport strategy) Suppose $W \subseteq \mathcal{L}_n$ s.t.

1) $\forall \Lambda \in \mathcal{L}_n$, $\overline{A\Lambda} \cap W \neq \emptyset$.

2) $\forall \Lambda \in W$, $\text{covrad}(\Lambda) = \max_{x\in\mathbb{R}^n} \min_{y\in\Lambda} \|x-y\| \leq \sqrt{n}/2$

then Minkowski (I') holds in dimension n. $SO_2(\mathbb{R}) \backslash SL_2(\mathbb{R}) / SL_2(\mathbb{Z}) \equiv$ 

We proved this last week. Recall the proof for $n=2$ using

Bottom ⌒ <=> WR

Bottom ⌒ <=> Stable

<u>Def.</u> Λ is <u>Well-rounded</u> if the set of shortest nonzero vectors in Λ span $\mathbb{R}^n$.

The set of Well-rounded lattices is denoted by WR. (<u>example:</u> $\mathbb{Z}^n \in WR$).

<u>Def.</u> For $\Lambda_0 \subseteq \Lambda$ subgroup, $|\Lambda_0| = \text{covol}(\text{span}_{\mathbb{R}}(\Lambda_0)/\Lambda_0)$ (w.r.t. restriction of euclidean norm)

$\Lambda \in \mathcal{L}_n$ is <u>stable</u> if $|\Lambda_0| \geq 1$ for all $\Lambda_0 \subseteq \Lambda$. <u>Example</u>: $\begin{pmatrix} 1 & * & * \\ & \ddots & * \\ 0 & & 1 \end{pmatrix}\mathbb{Z}^n$.

<u>Thm</u> For $n \leq 9$, $\forall \Lambda \in WR$ have covrad $\leq \sqrt{n}/2$.
Woods - Hans-Gill - Raka - Katharia - Sehmi

<u>Thm</u> For $n \leq 7$, $\forall$ stable lattice Λ have covrad $\leq \sqrt{n}/2$.
Hans-Gill - Raka - Katharia - Dutour-Sikric - Schurmann - Vallentin

} Both very recent & clever results.

Thm. If $A\Lambda$ is bounded (i.e. $\overline{A\Lambda}$ compact) then $\overline{A\Lambda} \cap WR \neq \emptyset$. (A)
McMullen

<u>Thm.</u> For any Λ, $\overline{A\Lambda} \cap ST \neq \emptyset$ (B)
Shapira - Wiess

<u>Thm</u> If $A\Lambda$ isn't bounded $\exists d_1 + \dots + d_k = n$ s.t. $\overline{A\Lambda} \ni \begin{pmatrix} B^{(1)} & & * \\ & \ddots & \\ 0 & & B^{(k)} \end{pmatrix}\mathbb{Z}^n$, $B^{(i)} \in SL_{d_i}(\mathbb{R})$
Birch - Swinnerton-Dyer

where $B^{(i)}\mathbb{Z}^{d_i}$ has bounded orbit under corresponding diagonal group.

cor. If Mink. (I) false in dim $=n$ but true for dim $<n$, then we can find a counterexample with a bounded A-orbit.

Labesgue Theorem on Dimension

Any cover of $\mathbb{R}^n$ using uniformly bounded open sets have order at least n, where the order of a cover $\{U_\alpha \mid \alpha\in A\}$ is $\max_{x\in\mathbb{R}^n} \#\{\alpha \mid x\in U_\alpha\}-1$ - ie, $\exists x\in\mathbb{R}^n$ covered at least $n+1$ times.

Example In $\mathbb{R}^2$,  cover of order ~~1~~ 1 (but not bounded),  bounded of order 1 (but not uniformly),  uni. bounded of order 2.

This can easily be proved using Čech cohomology of S_n.

Sketch of pf. for (A) in dim. 3: Given Λ and $\varepsilon>0$ define $\lambda_1(\Lambda)$ length of shortest $\underbrace{\text{vector}}_{\neq 0}$, for $\delta>0$ $\mathrm{Min}_\delta(\Lambda)=\{v\in\Lambda\setminus\{0\} \mid \|v\|\le\lambda_1(\Lambda)\cdot(1+\delta)\}$, $\dim_\delta(\Lambda)=\dim\operatorname{span}\mathrm{Min}_\delta(\Lambda)$, $\underset{1\le j\le n}{U_j^{(\varepsilon)}}=\{a\in A \mid \forall\delta \text{ in some nbd. of } j\varepsilon,\ \dim_\delta(a\Lambda)=j\}$.

McMullen shows $U_j^{(\varepsilon)}$ cover $A\cong\mathbb{R}^2$. We show the assumption of the Lebesgue thm. for conn. components of $U_1^{(\varepsilon)}, U_2^{(\varepsilon)}, U_3^{(\varepsilon)}$ and it would follow that some $a\in A$ is covered by 3 sets so $a\in U_{1,2,3}^{(\varepsilon)}$ and a has 3 shortest vectors of length $[\lambda_1(a\Lambda), \lambda_1(a\Lambda)(1+3\varepsilon)]$. Apply the argument for a_k $(\varepsilon=1/k)$. $a_k\Lambda$ is a sequence in a bounded set, and passing to a subsequence we'd get 3 lin. ind shortest vectors of equal length $\lim_{k\to\infty} a_k\Lambda\to\Lambda'\in$ WB (lin. ind as the angle between shortest vectors can't be $\searrow 0$).

Let $V\subset U_1^{(\varepsilon)}$ be a conn. comp. (clearly it's open). Let $a\in V$, $x=ay\in a\Lambda$ the shortest vector ($y\in\Lambda$). If one $\underset{(\text{say } i=1)}{x_i=0}$ we can apply $a_t=\begin{pmatrix} e^{2t} & & \\ & e^{-t} & \\ & & e^{-t}\end{pmatrix}$ so $a_t x_1\to 0$ and thus $a_t\Lambda\to\infty$ not bounded. $\max|x_i|$ must be bounded below uniformly. By $U_1^{(\varepsilon)}$'s definition, any $a\in U_1^{(\varepsilon)}$, $a\Lambda$ does not have two lin. ind. vectors of length in $[\lambda_1(a\Lambda), \lambda_1(a\Lambda)(1+\frac{3}{2}\varepsilon)]$. In particular it would stay $\pm ay$. So $V\subseteq\{a\in A \mid c_1\le\|ay\|\le c_2\}$ which is uni. bounded.

A similar approach works for $U_2^{(\varepsilon)}$ (and $U_3^{(\varepsilon)}$ - but anyway we just want to show it's nonempty). In higher dim those V won't be uni. bdd, but McMullen proves Covering Thm. U cover by open sets of $\mathbb{R}^n$ which is loc. fin.

and such that ($\exists R>0$) for any k distinct elements of $\mathcal{U}$, any conn. comp. of their intersection is $(R, n-k)$ affine. Then order$(\mathcal{U}) \geq n$.

Loc. fin - any compact set intersect finitely many elements of $\mathcal{U}$.

$X \subseteq \mathbb{R}^n$ is (R, j)-affine if it's contained within distance R of some affine subspace of dimension j.

Meaning in $\mathbb{R}^2$ - $k=1$ conn. components of elements are at bdd. distance from lines.
$k=2$ " " intersections at bdd. distance from points.

 works for $k=1$ but not 2,  won't work for $k=1$.

We'll (Michael Levin) only prove in dim. 2 - counterexample → counterexample to Lebesgue.

Let E be a conn. comp. of a set $\in \mathcal{U}$. Let $X_1, X_2, \dots$ be an aux. cover of the "strip" $\supseteq E$ with no 3-fold intersections by uni. bdd. sets. Let $X_i \cap X_j$ be big enough so that $\exists r>0\ \forall x \in X_i \cap X_j$, $B(x,r) \subseteq X_i$ or $B(x,r) \subseteq X_j$, $r >$ diam. of 2-fold intersections of sets in $\mathcal{U}$.

If there's an intersection one (say X_1) has to contain D completely, and we can modify X_2 so that the intersection won't ~~contain~~ intersect D. Continue "inductively".

In n dim, take care of n-fold int., $n-1$, $n-2$, ... would prove (A).

For (B) a similar idea works defining $\lambda^*(\Lambda) = \min_{\Lambda_0 \in \Lambda} |\Lambda_0|^{1/\mathrm{rank}(\Lambda_0)}$, $\mathrm{Min}^*_\delta(\Lambda) = \{\Lambda_0 \subseteq \Lambda \mid |\Lambda_0|^{1/\mathrm{rank}(\Lambda_0)} \in [\lambda^*(\Lambda), \lambda^*(\Lambda)(1+\delta)]\}$, $\dim^*_\delta = \dim \bigcup \mathrm{Min}^*_\delta(\Lambda)$, $U_j^{*(\varepsilon)} = \{a \in A \mid \forall \delta$ in a nbd. of $j\varepsilon$, $\dim^*_\delta(a\Lambda) = j\}$ is an open covering satisfying the assumptions of the covering thm., limits of $a_k \in U_n^{(1/k)}$ $k \to \infty$ are stable.

<u>Prop.</u> to prove covrad $\leq \sqrt{n}/2$ for stable lattices enough to check local maxima of covrad: $\mathcal{L}_n \to \mathbb{R}^n$.

<u>Thm</u> Eq. condition, "inhom. eutactic" = "perfect". So, it's enough to check finitely many examples in each dim - non in dim ≤ 5, 1 in dim. 6, 8 in dim. 7, ≈ 30 in dim 8, $\approx 2\cdot 10^6$ in dim 9. For the 9 local maxima in dim ≤ 7, all satisfy covrad$(\Lambda) \leq \sqrt{n}/2$. Probably possible to do that in dim 8, completely unfeasable in dim. 9.