W1202
12/1/15

Geometry of Numbers

Ex. sheet due date - end of break (flexible).

Minkowski's Conjecture

Part I - $\forall \Lambda \in \mathcal{L}_n\ \forall x \in \mathbb{R}^n\ \exists y \in \Lambda$ s.t. $\left|\prod_{i=1}^{n}(x_i - y_i)\right| \leq 1/2^n$.

Geom. Interpretation - $S = \{z \in \mathbb{R}^n \mid |\prod_{i=1}^{n} z_i| \leq 1/2^n\}$ then $\forall \Lambda \in \mathcal{L}_n$,

$\Lambda + S = \mathbb{R}^n$ (*)

Note that for $\Lambda = \mathbb{Z}^n$, $x = (1/2, 1/2, \dots, 1/2)$ we can't do any better than $1/2^n$.

Define $N: \mathbb{R}^n \to \mathbb{R}$: $N(x) = \left|\prod_{i=1}^{n} x_i\right|$. If $a \in A = \{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ diagonal of $\det = 1\}$, $N(ax) = N(x)$. So, $aS = S$; Λ satisfies (*) $\iff a\Lambda$ does.

part II If $\Lambda \in \mathcal{L}_n$ and $\sup_{x \in \mathbb{R}^n} \inf_{y \in \Lambda} N(x-y) = 1/2^n$ then $\Lambda = a\mathbb{Z}^n$ for some $a \in A$. Part I says this is $\leq 1/2^n$.

$\underbrace{\text{This implies}}_{\text{(Part I)}}$: K num. field ($\deg_{\mathbb{Q}} k = n$), O_K ring of integers, $N^k(\alpha) = \prod_{i=1}^{n} \sigma_i(\alpha)$ (distinct $\mathbb{C}$-embeddings), D_k the ~~determ~~ discriminant of K. Then $\forall x \in k\ \exists y \in O_k$ s.t. $|N(x-y)| \leq \frac{\sqrt{D_k}}{2^n}$ (deduction - if k totally real consider geometric embedding of O_K).

History of The Problem:

solved dim.	Author	year
2	Minkowski	1896/1900
3	Remak-Davenport	1928
4	Dyson	1948
5	Skubenko	1976
6	McMullen	2004
7	Hans-Gill, Sehmi	2009
8	"	" 2011
9	Hans-Gill, Raka, Celika	2015+

Define $\mu_n = \inf\{t > 0 \mid \forall x \in \mathbb{R}^n, \Lambda \in \mathcal{L}_n\ \exists y \in \Lambda$ s.t. $N(x-y) \leq t\}$. Minkowski - $\mu_n \overset{(?)}{=} 1/2^n$.

Best known $\mu_n \leq 1/2^{n/2} = 1/(\sqrt{2})^n$ by Chebotarev ('34). Some improvement by Blichfeldt, Mondell and Bombieri (but same order of growth).

Dynamical Interpertation

Recall A grid Γ is a set of form $\Lambda + x$. Unimodular if $d(\Lambda) = 1$.

Space of uni. grids $\mathcal{G}_n = ASL_n(\mathbb{R}) / ASL_n(\mathbb{Z})$. Again, Affine $SL_n(\mathbb{R}) \ltimes \mathbb{R}^n$, $SL_n(\mathbb{Z}) \ltimes \mathbb{Z}^n$.

Chabauty $\iff$ quotient top.

$1 \to \mathbb{R}^n \xrightarrow{x \mapsto \text{translation by } x} ASL_n(\mathbb{R}) \xrightarrow{\varphi \mapsto d\varphi} SL_n(\mathbb{R}) \to 1$ descends to $\pi: \mathcal{G}_n \to \mathcal{L}_n$, $\Gamma \mapsto \Lambda = \Gamma - \Gamma$. Fiber of π:

$\pi^{-1}(\Lambda) = \{\Lambda + x\} \cong \mathbb{R}^n / \Lambda$ compact torus. Let ν be the $SL_n(\mathbb{R})$ inv. measure on $\mathcal{L}_n$,

$d\nu(\Lambda + x) = d\mu(\Lambda)\, d\mathrm{Vol}_{\mathbb{R}^n/\Lambda}(x)$ defines a finite $ASL_n(\mathbb{R})$ invariant measure on $\mathcal{G}_n$. Let $W(\delta) = \{\Gamma \in \mathcal{G}_n \mid \overline{B}(0,\delta) \cap \Gamma \neq \emptyset\}$.

<u>Prop.</u> Mink(I) $\iff$ $\forall \delta > 0$, $\Gamma \in \mathcal{G}_n$, $A\Gamma \cap W(\frac{\sqrt{n}}{2}) \neq \emptyset$.

<u>pf.</u> $\Leftarrow$ Given $\Lambda \in \mathcal{L}_n$, $x \in \mathbb{R}^n$, define $\Gamma = \Lambda - x$. By assumption $\exists a \in A$ s.t. $a\Gamma = a\Lambda - ax \in W(\sqrt{n}/2)$, so it contains z with $\|z\| \leq \sqrt{n}/2$.

Denote $z = ay - ax$, so $N(x-y) = N(z) = |\prod_{i=1}^n z_i|$. So,

$N(x-y)^{2/n} = (\prod_{i=1}^n |z_i|^2)^{1/n} \leq \frac{1}{n}(\sum_{i=1}^n |z_i|^2) \leq \frac{1}{n} \cdot \frac{n}{4} = 1/4$, so $N(x,y) \leq 1/2^n$.

$\Rightarrow$ Γ a grid. We seek $a \in A$ s.t. $a\Gamma \in W(\sqrt{n}/2)$. Suppose first $\exists (v_1, \dots, v_n) \in \Gamma$ with some 0 entry, WLOG $v_1 = 0$. Take $a = \mathrm{diag}(2^{n-1}, 1/2, \dots, 1/2)$.

Then $a^k v = \frac{1}{2^k} v \xrightarrow[k\to\infty]{} 0$ so it's in $W(\sqrt{n}/2)$ for large enough k.

If there is no such v, write $\Gamma = \Lambda - x$, let $y \in \Lambda$ s.t. $N(x-y) \leq 1/2^n$.

write $y - x = (v_1, \dots, v_n)$. Choose $a \in A$ s.t. $|a_1 v_1| = \dots = |a_n v_n| = N(x-y)^{1/n} \leq 1/2$

Then $\|a(x-y)\| = \sqrt{\sum_{i=1}^n (a_i v_i)^2} \leq \sqrt{n}/2$, so $a\Gamma \in W(\sqrt{n}/2)$.

<u>Remark</u> $W(\sqrt{n}/2)$ contains the image of $\mathcal{L}_n$. We'll see $\exists$ subset $B \subseteq \mathcal{L}_n$ st. $\pi^{-1}(B) \subseteq W(\sqrt{n}/2)$. Lattices $\Lambda \in B$ have the property that for any x, $(\Lambda + x) \cap \overline{B}(0, \sqrt{n}/2) \neq 0$. This means $(\Lambda, \overline{B}(0, \sqrt{n}/2))$ is a covering of $\mathbb{R}^n$ - $\mathbb{R}^n = \Lambda + B(0, \sqrt{n}/2)$.

We'll use the Remak-Davenport strategy - given $\Gamma = \Lambda + x \in \mathcal{G}_n$, find $a \in A$ s.t. $a\Lambda \in B$ (in $\mathcal{L}_n$) hence $a\Gamma \in W(\sqrt{n}/2)$. This approach, while being "wastefull", its the only one which ~~works~~ has worked for $n \geq 4$.

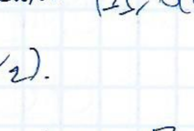
This gives a sufficient condition interms of the A-action on $\mathcal{L}_n$.

<u>Def</u> Given a lattice $\Lambda \subseteq \mathbb{R}^n$, $\mathrm{covrad}(\Lambda) = \inf\{r > 0 \mid (\Lambda, B(0,r))$ covers $\mathbb{R}^n\}$.

$\mathcal{U} = \{\Lambda \in \mathcal{L}_n \mid \mathrm{covrad}(\Lambda) \leq \sqrt{n}/2\}$, so if $\Gamma = \Lambda - x$, $A\Lambda \cap \mathcal{U} \neq \emptyset$ then Λ satisfies Mink(I).

<u>Proof of Mink I for $n=2$</u>

<u>Step I</u> Identify "$\mathcal{L}_n$ up to shape" $SO_2(\mathbb{R}) \backslash SL_2(\mathbb{R}) / SL_2(\mathbb{Z})$ =



<u>Step II</u> Draw $\mathcal{U}$ on picture - show in contains $\{z \in \mathbb{H} \mid |z| = 1\}$ to semi-circular arcs $\cup \mathbb{R}$ than

<u>Step III</u> Show A-orbits projects to Ω by $z \mapsto z \pm 1$, $z \mapsto -\frac{1}{z}$ (so it'll eventually hit $\mathcal{U}$).

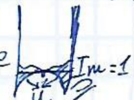
Given $\Lambda \in \mathcal{L}_n$ choose v_1, v_2 successive minima. Rotate Λ dilate ~~it~~ so that $v_1 = (1,0)$

$V_2 = (x,y)$ for $y>0$. $x^2+y^2 \geq 1$ since $\|V_2\| \geq \|V_1\|$. $|x| \leq \frac{1}{2}$ ~~=~~ $\|V_2 \pm V_1\|$ will be shorter, so $(x,y) \in \Omega$. Ties are given by φ_1, φ_2.

Short alg. - choose V_1, V_2. Think of them as $w_1, w_2 \in \mathbb{C}$, and mult. Λ by $1/w_1$ to bring V_1 to 1, so we map $SO_2(\mathbb{R})\Lambda$ to $\frac{w_2}{w_1}$.

For the proj. of $\mathcal{U}$, take $w = x+iy$ - the lattice in $\binom{1}{0}\mathbb{Z} + \binom{x}{y}\mathbb{Z}$.

The furthest points from any vertex are <u>deep point</u> of the fund. parallalogram. Such points are equidistant to 3 of $(0,0)$, $(1,0)$, (x,y), $(x+1,y)$. Assume first 3 so $x_0 = 1/2$ and then $x_0^2 + y_0^2 = (x-x_0)^2 + (y-y_0)^2$ so we get $x_0^2+y_0^2 = \frac{x^4+y^4+x^2+y^2+2x^2y^2-2x^3-2xy^2}{4y^2}$.

Since $d(\Lambda) = y$ the initial latte was scaled by $\sqrt{y}$. So the original $\mathrm{covrad}^2(\Lambda) = \frac{\text{same num.}}{4y^3}$. Computing explicitly we get some 

Why A-orbits $\to$ semicircles $\perp \mathbb{R}$? Take $\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\mathbb{Z}^2 = \binom{a}{c}\mathbb{Z} + \binom{b}{d}\mathbb{Z} \mapsto \frac{b+id}{a+ic}$. Think of $(a_1, a_2) = (e^t, e^{-t})$ - it acts by $a\Lambda = \binom{ae^t}{ce^{-t}}\mathbb{Z} + \binom{be^t}{de^{-t}}\mathbb{Z} \mapsto$

$$\frac{e^t b + ie^{-t}d}{e^t a + ie^{-t}c} = \frac{b+ie^{-2t}d}{a+ie^{-2t}c} = \begin{pmatrix} d & b \\ c & a \end{pmatrix} ie^{-2t}$$ as a Mobius transformation.

But ie^{2t} is $\mathrm{Im} \perp \mathbb{R}$ so it's mapped to a semicircle $\perp \mathbb{R}$.