W11D2
5/1/15

Geometry of Numbers

The covering problem - most "economical" way to cover $\mathbb{R}^n$ by translates of a body K?

Assume K Riemann measurable w. $Vol(K)>0$.

For Y discrete & countable, (K,Y) covering if $\mathbb{R}^n = K+Y$.

Periodic if $\exists \Lambda \subseteq \mathbb{R}^n$ lattice s.t. $Y=\bigcup_{i=1}^k x_i+\Lambda$, lattice ~~packing~~ covering if $Y=\Lambda$.

$\rho_+(K,Y)=\limsup_{r\to\infty} \sup_{x\in\mathbb{R}^n} \#\{y\in Y \mid x+[-r,r]^n \cap y+K \neq \emptyset\} Vol(K)/(2r)^n$. $\rho_-(K,Y)$ sim. for liminf, inf. Trivially $1\leq \rho_- \leq \rho_+$. Those are the upper & lower covering densities of (K,Y). The covering density of K is $\inf\{\rho_-(K,Y) \mid (K,Y) \text{ is a cover}\} = \Theta(K)$. Sim. define $\Theta_p(K)$ for periodic covering, $\Theta_L(K)$ for lattice covering.

Thm Suppose T inv. lin. trans. of $\mathbb{R}^n$. Then $\Theta, \Theta_p, \Theta_L$ are T-invariant.

and $\rho_+(K,Y)=\rho_-(K,Y)=\sum_{y\in Y}\frac{Vol(P\cap K+y)}{Vol(Y)} = \frac{N\, Vol\, K}{Vol\, P}$ for Y periodical, $Y=\bigcup_{i=1}^N x_i+\Lambda$

P fund. dom., ~~$N=\#\{y\in Y \mid P\cap(y+K)\neq\emptyset\}$~~. Also $\Theta(K)=\Theta_p(K)$.

In particular of lattice coverings, it's $Vol(K)/d(Y)$.

pf. Everything follows from $\Theta(K)=\Theta_p(K)$. Just take the non-periodic cover, a big box and duplicate it.

So from now on, for periodic coverings we write $\rho(K,Y)$.

Fact (From Mahler's compactness criterion) K compact - inf in def of Θ_L is min.

What is known about Θ, Θ_L? For $K_n = B_2(0,1) \subseteq \mathbb{R}^n$, $\Theta(K_2)=\Theta_L(K_2)$, $\Theta(K_3)=\Theta_L(K_3)$, $\Theta_L(K_4) \neq \Theta_L(K_5)$ all achieved by A_n^* from ex. 12 and only by it.

Conj $\Theta(K_n) > \Theta_L(\Theta_n)$ for $n\geq 5$.

In $n=6$ $\exists$ more efficient lattice then A_6^*.

Best known upper bounds - suppose K convex, $\Theta(K)\leq (o(1)+1)n\log n$, $\Theta_L(K)\leq n^{c\log\log n}$. If K=euclidean ball $\Theta_L(K)\leq c_1 n(\log n)^{c_2}$ $\left(c_2=\frac{\log_2 2\pi e}{2}\right)$.

We'll sketch a pf. for $\Theta_L(K)\leq c_1(n\log n)^{c_2}$.

steps of pf. Fix x, Λ. Let $\epsilon = \lim_{r\to\infty}\frac{Vol(B(0,r)\setminus K+\Lambda)}{Vol(B(0,r))} = \frac{Vol(P\setminus K+\Lambda)}{d(\Lambda)}$ for P fund. dom. First construct a cover in dim n_0

with $\varepsilon \le 1/2$, and inductively in $\dim = n_0 + k = n$ with $\varepsilon \searrow 0$ rapidly.

For very small ε, by inflating $K \times C_k$ we get a cover by balls.

Lemma 1 Let $1 \le k < n$, B ball in $\mathbb{R}^n$, then $B \supseteq B' \times C$ where B' ball in $\mathbb{R}^{n-k}$ and C cube, if $\frac{Vol(B' \times C)}{Vol(B)} \ge \left(\frac{4}{\pi}\right)^{k/2} \frac{(n-k)^{\frac{n-k}{2}}}{n^{n/2}} \frac{\Gamma(1+n/2)}{\Gamma\left(1+\frac{n-k}{2}\right)}$

pf. Direct computation.

Lemma 2 Let $\rho \in (0,1]$. $\exists$ lattice $\Lambda \subseteq \mathbb{R}^n$ s.t. $\rho(K,\Lambda) = \rho$, $\varepsilon(K,\Lambda) \le 1 - \rho + \rho^2/2$.

pf. WLOG $Vol(K) = \rho$. $\chi = \mathbb{1}_K$. For $\Lambda \subseteq \mathbb{R}^n$, $\varepsilon_\Lambda = \mathbb{1}_{\mathbb{R}^n \setminus K + \Lambda}$.

Define $\tau_\Lambda(x) = 1 - \sum_{u \in \Lambda} \chi(x-u) + \frac{1}{2} \sum_{\substack{u,v \in \Lambda \\ u \ne v}} \chi(x-u)\chi(x-v)$. Claim $\varepsilon_\Lambda(x) \le \tau_\Lambda(x)$.

This is since $\#\{u \in \Lambda \mid K + u \ni x\} = k$, then $\tau_\Lambda(x) = 1 - k + \frac{k(k-1)}{2} = \frac{1}{2}(k-1)(k-2)$. So, if $k=0$ $\varepsilon_\Lambda(x) = 1 = \tau_\Lambda(x)$, and if $k \ge 1$, $\varepsilon_\Lambda(x) = 0 \le \tau_\Lambda(x)$. Now $\varepsilon(K,\Lambda) = \int_{\mathbb{R}^n} \frac{\varepsilon_\Lambda(x)\, dVol(x)}{Vol\, P} \le \int_P \tau_\Lambda(x)\, dx = \int_P \Big[1 - \sum_{u \in \Lambda} \chi(x-u) + \frac{1}{2} \sum_{\substack{u,v \in \Lambda \\ u \ne v}} \chi(x-u)\chi(x-v) =$

$$1 - \sum_{u \in \Lambda} \int_P \chi(y)\, dVol\, y + \frac{1}{2} \sum_{\substack{u,v \in \Lambda \\ u \ne v}} \int_{P-u} \sum_{u_1 \in \Lambda \setminus \{0\}} \chi(y)\chi(y-u_1) =$$

$$1 - \int_{\mathbb{R}^n} \chi(y)\, dVol\, y + \frac{1}{2} \int_{\mathbb{R}^n} \sum_{u \in V \setminus \{0\}} \chi(y)\chi(y-u)\, dVol(y) \quad (*).$$ Define

$F(z) = \int_{\mathbb{R}^n} \chi(y)\chi(y-z)\, dVol(y)$ Then $(*) = 1 - \rho + \frac{1}{2} \int_{\mathbb{R}^n} F\, dVol =$

$$1 - \rho + \frac{1}{2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \chi(y)\chi(z-y)\, dy\, dz = \rho^2/2 - \rho + 1.$$

Lemma 3 Let $Vol(H) > 0$, $C = [0,2] \subseteq \mathbb{R}$. $K = H \times C$. $\exists$ lat $\Lambda_1 \subseteq \mathbb{R}^n$ s.t.

$$\rho(K, \Lambda_1) \le 2\rho(H,\Lambda), \quad \varepsilon(K,\Lambda_1) \le \varepsilon^2(K,\Lambda).$$

pf. Take $\mathbb{Z}^n, \mathbb{Z}^{n+1}$. For each $p \in \mathbb{Z}$, $S_p = \{x \in \mathbb{R}^{n+1} \mid p \le x_{n+1} < p+1\}$. $K + \Lambda_1 \cap S_p$ is obtained from $K + \Lambda_1 \cap S_0$ by translation $p(u + e_{n+1})$.

It's enough to understand $S_1 \cap (K + \Lambda_1)$. They have form $K + (v,0)$, $v \in \mathbb{Z}^n$, or $K + (v+u, 1)$. But the intersections are $(H+v) \times C'$ or $(H+v+u) \times C'$ for $C' = [1,2]$. Let $\mathbb{1}_{K'} = \chi$.

The set of points in $(S_1 \cap K + \Lambda_1)$ is of density $\le \varepsilon^2$.

Cor. $k \in \{1, \dots, n-2\}$, H as before, C_k k-dim. cube $K = H \times C_k \subseteq \mathbb{R}^n$.

Then $\exists \Lambda \subseteq \mathbb{R}^n$ s.t. $\rho(K,\Lambda) \le 2^k$, $\varepsilon(K,\Lambda) \le (1/2)^{2^k}$.

Lemma 4 Assume K convex, $h \in \mathbb{N}$, $\Lambda \subseteq \mathbb{R}^n$ s.t. $\varepsilon(K,\Lambda) \le \frac{1}{1+h^n}$. Then

$H = (1+\frac{1}{h})K$ has $\rho(H,\Lambda) = (1+\frac{1}{h})^n \rho(K,\Lambda)$ and $H+\Lambda = \mathbb{R}^n$.

pf. Let $\Lambda = \mathbb{Z}^n$ WLOG, $\Lambda_1 = h\Lambda$ sublattice of index h^n. Let $\Lambda_1, \dots, \Lambda_{h^n}$ be cosets, $C_h = [0,h)^n$, $S_i = C_h \cap (K+\Lambda_i)$. Then $\sum_{i=1}^{h^n} Vol(S_i) \geq h^n(1-\varepsilon(K,\Lambda))$. therefore $\exists j$, $Vol(S_j) \geq 1-\varepsilon(K,\Lambda) \geq \frac{h^n}{1+h^n}$. But cosets differ by translation they have same vol, so $Vol(S_1) \geq \frac{h^n}{1+h^n}$. Consider $\Lambda - \frac{1}{h}K$. This set is $\mathbb{Z}$-periodic in each coordinate. Let $C = (-1,0]^n$. So $C \cap (-\frac{1}{h}K+\Lambda)$ in $\frac{-1}{h}S_1$. Let $x \in \mathbb{R}^n$, $Vol(C_1 \cap (x-\Lambda - \frac{1}{h}K)) > \frac{1}{1+h^n}$ so these sets must intersect $\Rightarrow x = k_1 + \frac{k_2}{h} + \lambda$ for $k_1, k_2 \in K$, $\lambda \in \Lambda$, and by convexity $k_1 + k_2/h \in (1+1/h)K$.