W1 OD2
29/12/14

Geo. of Numbers

<u>Reminder</u>: $f: \mathbb{R}^n \to \mathbb{R}$ Riemann integrable, then $\hat{f}: \mathcal{L}_n \to \mathbb{R}$,

$$\hat{f}(\Lambda) = \sum_{u \in \Lambda \setminus \{0\}} f(u). \text{ Then } \mu(\Omega) \int_{\mathbb{R}^n} f \, d\mathrm{Leb}_{\mathbb{R}^n} = \int_{\mathcal{L}_n} \hat{f} \, d\nu.$$

μ Haar measure, Ω fund. dom of $\mathcal{L}_n = SL_n(\mathbb{R}) / SL_n(\mathbb{Z})$.
$\nu(A) = \mu(\Omega \cap \pi^{-1}$

Written differently, $\bar{\nu}(A) = \frac{\nu(A)}{\mu(\Omega)}$ is a prob. measure with $\int_{\mathbb{R}^n} f \, d\mathrm{Leb}_{\mathbb{R}^n} = \int_{\mathcal{L}_n} \hat{f} \, d\bar{\nu}$.

<u>Completion of pf.</u> Let $\chi = \chi_M$, $M \subseteq \mathbb{R}^n$ Riemman measurable, we claimed $\forall g \in SL_n(\mathbb{R}) \, \forall \varepsilon > 0$, $\varepsilon^n \sum_{u \in \mathbb{Z}^n \setminus \{0\}} g(\varepsilon g u) = O(\hat{\chi}_W)$ where $W = [-1, 1]^n$. The underlying const. depends only on M, n (not g, ε).

pf. Let $k = \# W \cap \Lambda$ where $\Lambda = g\mathbb{Z}^n$. By Minkowski I $k > 1$, so $k - 1 = \hat{\chi}_W \geq k/2$. For any cube $C = z + [-1/2, 1/2]$ then $\# C \cap [-1/2, 1/2]^n \leq k$, since the difference between any 2 such points is in $\Lambda \cap W$. By rescaling $C' = z + [-\varepsilon/2, \varepsilon/2]^n$ Then $C' \cap \varepsilon \Lambda \leq k$.
Suppose $M \subseteq [-\rho, \rho]^n$ and let $q = \lfloor \frac{2\rho}{\varepsilon} \rfloor + 1$. Then $\varepsilon\Lambda$ has at most $k \cdot q^n$ point in $[-\rho, \rho]^n$ (can cover by q^n C' cubes).
So, $\sum_{u \in \mathbb{Z}^n \setminus \{0\}} \chi(\varepsilon g u) \leq k \left(1 + \lfloor \frac{2\rho}{\varepsilon} \rfloor\right)^n \leq k \left(\frac{1 + 2\rho}{\varepsilon}\right)^n \Rightarrow \varepsilon^n \sum_{u \in \mathbb{Z}^n \setminus \{0\}} \chi(\varepsilon g u) \leq (2\rho + 1)^n k \leq$

$2(2\rho + 1)^n \hat{\chi}_W(\Lambda)$.

<u>Another point of View (+ generalizations):</u>

$f \in \mathrm{Map}(\mathbb{R}^n, \mathbb{R}) \longmapsto \hat{f} \in \mathrm{Map}(\mathcal{L}_n, \mathbb{R})$.

$f \geq 0 \Rightarrow \hat{f} \geq 0$. $G = SL_n(\mathbb{R})$ acts by precomposition on these two spaces. $f \mapsto \hat{f}$ is equivariant (that is, $\hat{f} \circ g = \widehat{f \circ g}$).
Now, let ν be any G-inv. measure on $\mathcal{L}_n$ such that $f \in C_c(\mathbb{R}^n)$, $\hat{f} \in L^1(\nu)$, then $f \longmapsto \int_{\mathcal{L}} \hat{f} \, d\nu$ is a positive linear functional on $C_c(\mathbb{R}^n)$. By Riesz representation thm. this assignment is given by integration on $\mathbb{R}^n$. By equivariance it's $SL_n(\mathbb{R})$.

<u>Riesz Rep. Thm.</u>: X loc. comp, matrizable, $\exists$ bijection between

pos. lin. functionals on $C_c(X)$ and loc. fin. reg. Borel measures.
But the only such measures ar $Leb_{\mathbb{R}^n}$ and $\delta_0(A)=\begin{cases}1 & 0\in A\\ 0 & o/w\end{cases}$.
Therefore $\int_{\mathcal{L}_n} \hat{f}\, d\nu = c_1 f(0) + c_2 \int_{\mathbb{R}^n} f \, dLeb_{\mathbb{R}^n}$. Assume we can switch the order of summation & integration in $\int_{\mathcal{L}_n} \hat{f} d\nu = \int_{\mathcal{L}_n} \sum_{u\in\Lambda\setminus\{0\}} f(u) d\nu(\Lambda)$.
[Note $\hat{f}$ isn't neccessarily compactly supported or even bounded. This can be achieved by dominated convergence & claim from the beginning of the theorem. Then if we take functions with smaller and smaller support $\int_{\mathbb{R}^n} f_k \, dLeb_{\mathbb{R}^n} = 1$, $f_k \geq 0$, $f_n(0)=\max_{\mathbb{R}^n} f = 1$, vanishes outside $B(0, 1/k)$. Then $\hat{f}_k(\Lambda) \to 0$ for each fixed Λ, we'd get $0 = \lim_{k\to\infty} \int_{\mathcal{L}_n} f_k \, d\nu = c_1$.

Two tricks for showing $c_2 = 1$ - Let $f \in C_c(\mathbb{R}^n)$, and $\varepsilon > 0$.
Set $f_\varepsilon(x) = f(\varepsilon x)$. Then $c_2 \varepsilon^n \int_{\mathbb{R}^n} f \, dLeb_{\mathbb{R}^n} = \int \hat{f}_\varepsilon \, d\nu$, so
$c_2 \int_{\mathbb{R}^n} f \, dLeb_{\mathbb{R}^n} = \underbrace{\int_{\mathcal{L}_n} \varepsilon^n \sum_{u\in\Lambda\setminus\{0\}} f(\varepsilon u) \, d\nu}_{\downarrow \varepsilon\searrow 0 \;\; \int_{\mathbb{R}^n} f \, dLeb_{\mathbb{R}^n}}$ so $c_2 = 1$.

Alternatively, each Λ has asymptotic density 1. Let $f = \chi_{B(0,1)}$, so $f_\varepsilon = \chi_{B(0,1/\varepsilon)}$. Then for each Λ,
$\varepsilon^n \hat{f}_\varepsilon(\Lambda) = \varepsilon^n \#((\Lambda\setminus\{0\}) \cap B(0, 1/\varepsilon)) \xrightarrow[\varepsilon\to 0]{} V_n$ and we get
$V_n = \lim_{\varepsilon\to 0} \int_{\mathcal{L}} \varepsilon^n f_\varepsilon(\Lambda) d\nu = \lim_{\varepsilon\to 0} \varepsilon^n c_2 \int_{\mathbb{R}^n} f_\varepsilon \, dLeb_{\mathbb{R}^n} = c_2 V_n$.

Generalizations

Techical points - $\hat{f} \in L^1(\nu)$, can change $\lim \int$.
We didn't use $\mathcal{L}_n$'s structure, so a similar argument works similarly for, say, sets of asymptotic density 1.
X, Y with a G-action on both, and a map $Y \to Sub(X)$, $y \mapsto \Lambda_y$ G-eq. and for $f: X \to \mathbb{R}$, $\hat{f}(y) = \sum_{x\in\Lambda_y} f(y)$ X - Need to know all G-inv. measures loc. fin. Borel measure. Y - $\exists$ invariant prob. measure ν.
Works of Siegel, Weil, Veach, Margulis, ...

Recall Packing Problem: $\Delta(K) = \inf\{d(\Lambda) \mid \Lambda \subset \mathbb{R}^n \text{ lattice},\ (V\setminus\{0\}) \cap K \neq \emptyset\}$.

Minkowski I K centrally symm. & convex $\Leftarrow\Rightarrow Vol(K) \leq 2^n \Delta(K)$, or
$c(K) := \frac{Vol(K)}{\Delta(K)} \leq 2^n$. c is invariant to rescaling.

Cor K bdd. & Riemann measurable then $c(K) \geq 1$. Assume WLOG $Vol(K) < 1$ and let $\chi = \chi_K$. Then $1 > Vol(K) = \int_{\mathbb{R}^n} \chi \, dVol = \int_{\mathcal{L}_n} \hat{\chi} \, d\bar{\nu}$ so $\exists \Lambda \in \mathcal{L}_n$ s.t. $\hat{\chi}(\Lambda) = \# K \cap (\Lambda \setminus \{0\}) < 1$, so $\hat{\chi}(\Lambda) = 0$ and $\Delta(K) \leq 1$. If we rescale so that $Vol(K) \nearrow 1$, $c(K) = \frac{Vol(K)}{\Delta(K)} \geq 1$.

Def K is a [sym.] star body if it's compact & $\forall x \in K$ $t \in [0,1)$ [$|t|<1$] $tx \in \text{int } K$.

Cor If K is a [sym.] star body in $\mathbb{R}^n$, $c(K) \geq [2 \cdot] \zeta(n)$.

pf. Let $\chi = \chi_K$, $\hat{\hat{f}}(\Lambda) = \sum_{u \text{ primitive in } \Lambda} f(u)$. Since $\Lambda = \{0\} \cup P(\Lambda) \cup 2P(\Lambda) \cup \ldots$, we get $\int_{\mathcal{L}_n} \hat{\hat{f}}(\Lambda) d\bar{\nu} = \frac{1}{\zeta(n)} \int_{\mathbb{R}^n} f$. Observe $\hat{f}(\Lambda) > 0$ then $\hat{f}(\Lambda) \geq 1$ before so $\hat{\hat{f}}(\Lambda) \geq 1$ [2] since K is [sym.] star body. Continue as before.

For $K = B(0,1)$ write $c_n = c(K)$. We just proved $2\zeta(n) \leq c_n \leq 2^n$. This (for sym. star body) is the Minkowsk-Hlawka theorem.

Note $c_n = Vol$(largest elipsoid in $\mathbb{R}^n$ around 0 whose int. is disjoint from $\mathbb{Z}^n \setminus \{0\}$).

Goal Improve bounds for c_n. Currently the best upper bounds are of the form $c_n \leq (1-\delta)^n$ for some $\delta > 1$, lower - $c_n \geq 0.73 n$ for all large enough n [Rogers '47], $c_n \geq 2(n-1)$ for any n [K. Ball, '92], Krivelevich, Citsyn & Vardi proved better bounds for some n values in '04, and [S. Vance '11] $c_n > 2.2n$ when $4|n$. Thm $\exists n_k \nearrow \infty$ s.t. $c_{n_k} \geq \frac{1}{2} n_k \log\log n_k$. Also, for all large enough n, $c_n \geq 65963$ $\left[2\sinh^2(\pi e)/\pi e^3\right]$ [Venkatesh '12].

Sketch of pf. $k = \mathbb{Q}(\zeta_n)$ (ζ_n prim. nth root of unity), $\deg k/\mathbb{Q} = \phi(n)$, $V = k^2$ and let $V_{\mathbb{R}} = V \otimes_{\mathbb{Q}} \mathbb{R}$, so that $\dim_{\mathbb{Q}} V = \dim_{\mathbb{R}} V_{\mathbb{R}} = 2\phi(n)$. Let $\mathcal{O}$ be the ring of integers in k, $\Lambda_0 = \mathcal{O}^2 \subseteq V_{\mathbb{R}}$ a lattice. Mult. by ζ_n preserves Λ_0.

So, $x \in \Lambda_0 \setminus \{0\} \Rightarrow x, \xi_n x, \ldots, \xi_n^{n-1} x$ are distinct elements of Λ_0. For any fin. group acting on a vector space V, $\exists$ pos. def. G-inv. form q_0 on V (by avg.). So take q_0 inv. under mult. by ξ_n so that (*) $\underbrace{x, \xi_n x, \ldots}_{\in \Lambda_0}$ have equal q_0-length.

Define $G = SL_2(k \otimes_{\mathbb{Q}} \mathbb{R}) \circlearrowright V_{\mathbb{R}}$ commutes with $\xi_n \cdot -$, so any lattice in $G\Lambda_0$ has (*). $\mathrm{Stab}\,\Lambda_0 = SL_2(\mathcal{O})$ = a lattice in G.

Now $G\Lambda_0 \cong G/SL_2(\mathcal{O}) = Y$, $\exists\, \bar{\nu}$ G-inv. $\overset{\text{prob.}}{\frown}$ measure on Y, so that $\int_{\in \Lambda_0} \hat{f}(\Lambda)\, d\bar{\nu} = \int_{V_{\mathbb{R}}} f \, d\mathrm{Leb}$ (after normalizing). Now,

we get $C_{2\phi(n)} \geq n$ by a similar argument. If $n = \prod_{\substack{p<x \\ p \text{ prime}}} p$

we get this result.