w8D1 29/04/2012 הרצאה

שני מסילות $\gamma:[a,b]\to\mathbb{C}$, C^1_{pw}
$\gamma_1:[c,d]\to\mathbb{C}$

נקראות שקולות, $\gamma\sim\gamma_1$
אם קיים
$\varphi:[c,d]\xrightarrow{\text{על}}[a,b]$

- $\varphi\in C^1_{pw}$, $\varphi\nearrow$ (מונוטונית עולה)

כך ש $\gamma_1=\gamma\circ\varphi$

$$\int_\gamma f(z)\,dz=\int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

- $\gamma_1\sim\gamma\to\int_{\gamma_1}f(z)\,dz=\int_\gamma f(z)\,dz$

$$L(\gamma)=\int_a^b|\gamma'(t)|\,dt$$

$$L(\gamma_1)=\int_c^d|\gamma_1'(s)|\,ds=\int_c^d|\gamma'(\varphi(s))|\,\varphi'(s)\,ds=\int_a^b|\gamma'(t)|\,dt=L(\gamma)$$

$$\boxed{L(\gamma_1)=L(\gamma)}$$

הפוך / חיבור / חיסור מסילות:

$\gamma : [a,b] \to \mathbb{C}$, $a < c < b$

$\Rightarrow \gamma_1 = \gamma|_{[a,c]}$, $\gamma_2 = \gamma|_{[c,b]}$

$$\gamma := \gamma_1 + \gamma_2$$

$$\int_{\gamma_1 + \dots + \gamma_n} f(z)\,dz = \sum_{i=1}^{n} \int_{\gamma_i} f(z)\,dz$$

- הפוך:

$$\int_{-\gamma} f(z)\,dz = -\int_{\gamma} f(z)\,dz$$

$$-\gamma : [-b,-a] \to \mathbb{C}$$

$$-\gamma(t) = \gamma(-t)$$

- חסם:

$$\left|\int_{\gamma} f(z)\,dz\right| \le L(\gamma)\cdot \max_{z\in\gamma} |f(z)|$$

הוכחה:

$$\left|\int_{\gamma} f(z)\,dz\right| = \left|\int_a^b f(\gamma(t))\,\gamma'(t)\,dt\right| \le \max_{z\in\gamma}|f(z)| \int_a^b |\gamma'(t)|\,dt$$

מסקנה: $f \in A(G)$, $f' \in C(G)$

וכן $\gamma \subseteq G$ כך ש $\gamma(a) = z_0$, $\gamma(b) = z_1$
אזי:

$$\int_\gamma f'(z)\,dz = f(z_1) - f(z_0)$$

בפרט, אם γ סגורה:

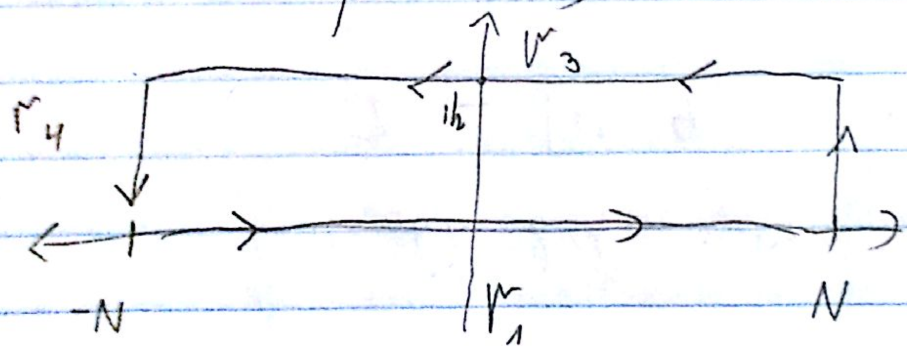
$$\int_\gamma f'(z)\,dz = 0$$

<u>דוגמה</u>

$$f(z) = e^{-z^2}$$

$$\exists\, F \in A(\mathbb{C}) \,.\, F'(z) = f(z)$$

נסמן Γ מסילת מלבן:



- נפצל ל- 4 מסילות:

$$\Gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$$

זו מסילה סגורה, F אנליטית ולכן:

$$\int_\Gamma f(z)\,dz = 0$$

- נחשב:

$\gamma_2 : [0,h] \to \mathbb{C}$

$\gamma_2(t) = N + it$

$(N+it)^2 = N^2 - t^2 + 2iNt$

$$\left|e^{-(N+it)^2}\right| = e^{t^2-N^2} \le e^{h^2-N^2}$$

$$\left|\int_{\gamma_2} f(z)\,dz\right| \le e^{h^2-N^2}\cdot L(\gamma_2) = e^{-N^2}e^{h^2}\cdot h \xrightarrow[N\to\infty]{} 0$$

בצורה דומה:

$\gamma_4(t) = -N+it$

$$\int_{\gamma_4} f(z)\,dz \xrightarrow[N\to\infty]{} 0$$

מצד שני:

$\gamma_1 : [-N,N] \to \mathbb{C}$

$\gamma_1(t) = t$

$$\int_{\gamma_1} e^{-z^2}dz = \int_{-N}^{N} e^{-t^2}dt \xrightarrow[N\to\infty]{} \int_{-\infty}^{\infty} e^{-t^2}dt = \sqrt{\pi}$$

$-\gamma_3 : [-N,N] \to \mathbb{C}$

$-\gamma_3(t) = t + ih$

$$-\int_{\gamma_3} f(z)\,dz = \int_{-N}^{N} e^{-(t+ih)^2} = e^{h^2}\int_{-N}^{N} e^{-t^2}(\cos 2th + i\sin 2th)\,dt$$

ונקבל:

$$\int_{-\Gamma^3} f(z)dz = \int_{\Gamma_1} f(z)dz - \int_{\Gamma_2+\Gamma_4} f(z)dz \to \int_{-\infty}^{\infty} e^{-t^2}dt$$

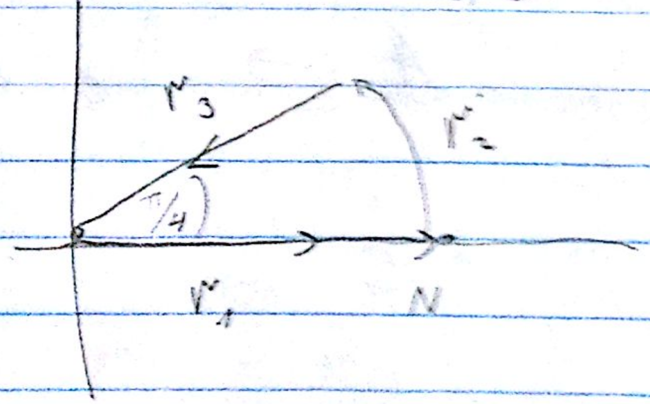
פה:

$$e^{h^2}\int_{-\infty}^{\infty} e^{-t^2}\cos 2th\, dt = \int_{-\infty}^{\infty} e^{-t^2}dt$$

$$\int_{-\infty}^{\infty} e^{-t^2}\cos 2th\, dt = e^{-h^2}\sqrt{\pi}$$

2. ניקח $f(z) = e^{-z^2}$ אזי

נסתכל על המסלול:



$$\int_{\Gamma_1} e^{-z^2}dz = \int_0^N e^{-t^2}dt \xrightarrow[N\to\infty]{} \frac{\sqrt{\pi}}{2}$$

$-\Gamma_3(t) = te^{i\pi/4}$

$$-\int_{\Gamma_3} e^{-z^2}\,dz = \cancel{\int_0^N e^{-t^2 i} \cdot e^{i\pi/2}\,dt}$$

$$= \int_0^N e^{-(te^{i\pi/4})^2} \cdot e^{i\pi/4}\,dt =$$

$$= \int_0^N e^{-t^2 i} \cdot e^{i\pi/4}\,dt =$$

$$= \int_0^N (\cos t^2 - i\sin t^2)\left(\frac{\sqrt2}{2} + i\frac{\sqrt2}{2}\right)dt =$$

$$= \frac{\sqrt2}{2}\int_0^N \cos t^2 + \sin t^2\,dt + i\frac{\sqrt2}{2}\int_0^N \cos t^2 - \sin t^2\,dt$$

אם נכון: $\int_{\Gamma_2} f(z)\,dz \xrightarrow[N\to\infty]{} 0$

נקבל:

$$\frac{\sqrt2}{2}\int_0^\infty \cos t^2 - \sin t^2\,dt + i\frac{\sqrt2}{2}\int_0^\infty \cos t^2 - \sin t^2\,dt = \frac{\sqrt\pi}{2} + i\cdot 0$$

$$\int_0^\infty \cos t^2\,dt - \int_0^\infty \sin t^2\,dt \Leftarrow \int_0^\infty \cos t^2 - \sin t^2\,dt = 0$$ ונקבל:

וכן נקבל

$$\sqrt2\int_0^\infty \cos t^2\,dt = \frac{\sqrt\pi}{2}$$

$$\Rightarrow \int_0^\infty \cos t^2\,dt = \int_0^\infty \sin t^2\,dt = \frac{\sqrt\pi}{2\sqrt2}$$

(נותר להוכיח לגבי Γ_2:

$$\Gamma_2 : [0, \tfrac{\pi}{4}] \to \mathbb{C}$$

$$\Gamma_2(t) = Ne^{it}$$

$$z^2 = N^2 e^{2it}$$
$$Re(z^2) = N^2 \cos 2t$$

$$|e^{-z^2}| = e^{-N^2 \cos 2t}$$

$$\Gamma_2'(t) = Nie^{it}$$

$$|\Gamma_2'(t)| = N$$

$$I_N = \left| \int_{\Gamma_2} e^{-z^2} dz \right| \le \int_0^{\pi/4} e^{-N^2 \cos 2t} \cdot N \, dt \le$$

$$\le N \int_0^{\pi/4} e^{-N^2 \cos 2t} dt \overset{s = \frac{\pi}{2} - 2t}{=} N \int_0^{\pi/2} e^{-N^2 \sin s} ds$$

$$\overset{\frac{2}{\pi}s \le \sin s}{\le} N \int_0^{\pi/2} e^{-N^2 \frac{2}{\pi} s} \underset{u = N^2 \frac{2}{\pi} s}{\le} \frac{N}{N^2 \frac{2}{\pi}} \int_0^{\infty} e^{-u} du = \frac{\pi}{2} \cdot \frac{1}{N} \xrightarrow[N \to \infty]{} 0$$

WBDS 03/05/2012 הרצאה

תזכורת:

$f(x,y)\geq 0$

$$\int_0^\infty\left(\int_0^\infty f(x,y)\,dx\right)dy = \sup_M \sup_N \int_0^M\left(\int_0^N f(x,y)\,dx\right)dy$$

- דרך לחשב

יהי $x>0$

$$J = \int_0^\infty e^{-t^2}\,dt \overset{t \to tx}{=} x\cdot\int_0^\infty e^{-t^2x^2}\,dt$$

$$J^2 = J\cdot\int_0^\infty e^{-t^2}\,dt = J\cdot\int_0^\infty e^{-x^2}\,dx = \int_0^\infty e^{-x^2} J\,dx =$$

$$= \int_0^\infty e^{-x^2}x\left(\int_0^\infty e^{-t^2x^2}\,dt\right)dx =$$

$$= \int_0^\infty\left(\int_0^\infty e^{-x^2(t^2+1)}x\,dx\right)dt = \left\{\begin{matrix} u = x^2(t^2+1) \\ du = 2x(t^2+1)\,dx\end{matrix}\right\} =$$

$$= \int_0^\infty\left(\int_0^\infty e^{-u}\cdot\frac{1}{2(t^2+1)}\,du\right)dt = \int_0^\infty\left(\frac{1}{2(t^2+1)}\right)dt =$$

$$= \frac{1}{2}\arctan t\Big|_0^\infty = \frac{\pi}{4}$$

$$J^2 = \frac{\pi}{4}, \quad \boxed{J = \sqrt{\pi}/2}$$