Complex Function Theory

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1 Power series

In the last lesson we talked about power series. We will finish this subject before moving on.

$$f(z) = \sum_{n\geq 0} a_n z^n \qquad R^{-1} = \limsup_{n\to\infty} |a_n|^{1/n}$$

In the specaial case of $R = 1$, $z = e^{i\theta}$ and the function is an analytical Fourier series which is defined for all natural n as opposed to the normal Fourier seires which is defined for all $n \in \mathbb{Z}$.

$$\sum_{n\geq 0} a_n e^{in\theta}$$

Theorem 1 (Abel). *Let $R = 1$ and $\sum_{n\geq 0} a_n$ converges then $f(z) \to \sum_{n\geq 0} a_n$, $z \to 1\, |1-z| \leq C\left(1-|z|\right)$.*

Theorem 2 (Tauber). *If $n|a_n| = o(1)$ as $n \to \infty$ and the limit $\lim_{x\uparrow 1} f(x)$ exists, then $\sum_{n\geq 0} a_n$ converges.*

Example 1. 1.

$$e^z = \sum_{n\geq 0} \frac{z^n}{n!} \qquad R = \infty \qquad e'(z) = e(z)$$

2.

$$\sin z = \sum_{m\geq 0} (-1)^m \frac{z^{2m+1}}{(2m+1)!} \qquad \cos z = \sum_{m\geq 0} (-1)^m \frac{z^{2m}}{(2m)!}$$

3.

$$\log(1+Z) = \sum_{n\geq 0} (-1)^n \frac{z^n}{n} \qquad R = 1 \qquad \Re(1+z) > 0 \qquad \log'(z) = \frac{1}{1+z}, \log(0) = 1$$

4.

$$z = \tan w = \frac{1}{i}\frac{e^{iw}-e^{-iw}}{e^{iw}+e^{-iw}} \Rightarrow w = \arctan z = \frac{1}{2i}\log\frac{1+iz}{1-iz} \qquad \Re\left(\frac{1+iz}{1-iz}\right) = \frac{1-|z|^2}{|1-iz|^2} > 0$$

$$(1+iz)(1+i\bar{z}) = 1 + i(z+\bar{z}) - |z|^2 \qquad \arctan z = \sum_{n\geq 1} (-1)^{n-1} \frac{z^{2n-1}}{2n-1}$$

5.

$$\sigma \in \mathbb{C}\backslash\mathbb{Z}_+ \qquad (1+z)^\sigma = \sum_{n\geq 0} \frac{\sigma\cdot(\sigma-1)\cdots(\sigma-n+1)}{n!} z^n = \binom{\sigma}{n}$$

2 Line integrals

[1] Let $f : I \to \mathbb{C}$ and $I = [a, b]$

$$\int_I f = \int_I \Re f + i \cdot \int_I \Im f$$

Note 1. Obviously, both $\Re f$ and $\Im f$ are integrable.

Claim 1.

$$\left| \int_I f \right| \leq \int_I |f|$$

Proof. We will assume that $\int_I f \neq 0$ and we will define

$$\lambda = \frac{\left| \int_I f \right|}{\int_I f} \qquad |\lambda| = 1$$

$$\left| \int_I f \right| = \lambda \int_I f = \overbrace{\int_I \lambda f}^{\in \mathbb{R}} = \int_I \Re (\lambda f) \leq \int_I |\lambda f| = \int_I |f|$$

□

2.1 Length

$\gamma : I = [a, b] \to \mathbb{C}$ is continuous and $\Pi = \{t_0 = a < t_i < t_N = b\}$ then

$$\gamma_\Pi = \bigcup_{j=1}^{N-1} \left[\gamma(t_j), \gamma(t_{j+1}) \right] \qquad L(\gamma_\Pi) = \sum_{j=0}^{N-1} \left| \gamma(t_{j+1}) - \gamma(t_j) \right|$$

Definition 1.

$$L(\gamma) = \sup_\Pi L(\gamma_\Pi)$$

If $L(\gamma) < \infty$ then γ is a rectifiable curve.

Theorem 3. *If γ is in C^1 then is is a rectifiable curve.* $L(\gamma) = \int_I |\dot{\gamma}|$

Proof. 1.

$$\Pi \succ \Pi' \Rightarrow L(\gamma_{\Pi'}) \geq L(\gamma_\Pi)$$

2. Additivity: $c \in (a, b) \Rightarrow L(\gamma) = L\left(\gamma|_{[a,c]}\right) + L\left(\gamma|_{[c,b]}\right)$.

- Let Π_1 be a partition of $[a, c]$ and Π_2 is a partition of $[c, b]$, $\Pi = (\Pi_1, \Pi_2)$

$$L(\gamma) \leq L(\gamma_\Pi) = L(\gamma_{\Pi_1}) + L(\gamma_{\Pi_2}) \Rightarrow L(\gamma) \geq L(\gamma_1) + L(\gamma_2)$$

- Π is a partitioin of $[a, b]$ we will choose (Π_1, Π_2) such that $\Pi \succ (\Pi_1, \Pi_2)$

$$\Rightarrow L(\gamma_\Pi) \leq L(\gamma_{\Pi_1}) + L(\gamma_{\Pi_2}) \leq L(\gamma_1) + L(\gamma_2) \Rightarrow L(\gamma) \leq L(\gamma_1) + L(\gamma_2)$$

[1] Also known as a conture integral or a curve integral

3. If γ is an arc C^1 then $L(\gamma) \leq \int_I |\dot{\gamma}|$

$$L(\gamma_\Pi) = \sum_{j=0}^{N-1} \left|\gamma(t_{j+1}) - \gamma(t_j)\right| = \sum_{j=0}^{N-1} \left| \int_{t_j}^{t_{j+1}\dot{\gamma}} \right| \leq \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} |\dot{\gamma}| = \int_I |\dot{\gamma}|$$

Since it is additive, it is sufficient to prove this for an arc C^1

4. $t \in [a,b]$ we will define $l(t) = L\left(\gamma|_{[a,t]}\right)$, $a \leq t' < t'' \leq b$

$$\frac{\left|\gamma(t'') - \gamma(t')\right|}{\underbrace{t'' - t'}_{|\dot{\gamma}(t)|}} \leq \frac{l(t'') - l(t')}{t'' - t'} = \frac{L\left(\gamma|_{[t',t'']}\right)}{t'' - t'} \leq \frac{1}{t'' - t'} \int_{t'}^{t''} |\dot{\gamma}| \to \left|\dot{\gamma}(t)\right|$$

$l(t)$ is differentiable and $\frac{dl}{dt} = \left|\dot{\gamma}(t)\right|$, $l(a) = 0 \Rightarrow l(t) = \int_a^t |\dot{\gamma}| \Rightarrow L(\gamma) = l(b) = \int_a^b |\dot{\gamma}|$.

□

- $y = f(x)$, $a \leq x \leq$

$$L(\gamma) \int_a^b \sqrt{1 + f'(x)^2} dx$$

-

$$\gamma = \left\{|z| = \mu = \mu(\theta), a \leq \theta \leq b\right\} \qquad L(\gamma) = \int_a^b \sqrt{\mu^2(\theta) + \mu'(\theta)^2} d\theta$$

$\gamma : I \to \mathbb{C}$, $f \in C\left(\gamma[a,b]\right)$ and is C^1

Definition 2.

$$\int_\gamma f(z) dz := \int_I f\left(\gamma(t)\right) \dot{\gamma}(t) \, dt$$

$c_1, c_2 \in \mathbb{C}$

$$\int_\gamma (c_1 f_1 + c_2 f_2) = c_1 \int_\gamma f_1 + c_2 \int_\gamma f_2$$

$$\int_{\gamma_1 + \cdots + \gamma_n} = \sum_{j=1}^{n} \int_{\gamma_j} f$$

$$\left(-\gamma(s)\right) := \gamma(b - s) \qquad 0 \leq s \leq b - a \qquad \int_{-\gamma} f = -\int_\gamma f$$

Claim 2 (Newton-Leibniz). Let $\gamma : I \to G$ where G is a domain and $f \in A(G), f' \in C(G)$ Then

$$\int_\gamma f'(z) \, dz = f\left(\gamma(b)\right) - f\left(\gamma(a)\right)$$

Proof.

$$\int_\gamma f' = \int_a^b \overbrace{f'\left(\gamma(t)\right) \dot{\gamma}(t) dt}^{= \frac{d}{dt}\left[f\left(\gamma(t)\right)\right]} \stackrel{\text{N-L}}{=} (f \circ \gamma)(b) - (f \circ \gamma)(a)$$

□

Corollary 1. *If γ is closed ($\gamma(b) = \gamma(a)$) then $\int_\gamma f' = 0$.*

Example 2. $\gamma(t) = z_0 + \rho e^{it}$ $\dot\gamma(t) = \rho i e^{it}$

$$\int_\gamma (z - z_0)^n := \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{cases}$$

$$(z - z_0)^n = \left(\frac{(z - z_0)^{n+1}}{n+1}\right)' \qquad \int_\gamma \frac{dz}{z - z_0} = \int_0^{2\pi} \frac{i\rho e^{it}}{\rho e^{it}} dt = 2\pi i$$

$$2^{2n} \int_0^{2\pi} \frac{1}{e^{it}} \left(\frac{e^{it} + e^{-it}}{2}\right)^{2n} i e^{it} dt = \int_\gamma \frac{1}{z}\left(z + \frac{1}{z}\right)^{2n} dz = \int_\gamma \binom{2n}{n} \frac{dz}{z} = \binom{2n}{n} \cdot 2\pi i$$

$$\int_0^{2\pi} \cos^{2n} t\, dt = 2\pi \cdot 2^{-2n} \binom{2n}{n}$$

$\gamma : I = [a, b] \to \mathbb{C}$ $t(s) \in C^1$

$$\tilde I = [\alpha, \beta] \xrightarrow[t(s)]{1-1 onto} [a, b] \xrightarrow{\gamma} \mathbb{C} \qquad \tilde\gamma(s) = \gamma(t)$$

Claim 3.

$$\int_\gamma f dz = \int_{\tilde I} f dz$$

Proof.

$$\int_{\tilde\gamma} f dz = \int_\alpha^\beta f\left(\tilde\gamma(s)\right) \dot{\tilde\gamma}(s) ds = \int_\alpha^\beta f\left(\gamma\left(t(s)\right)\right) \dot\gamma\left(t(s)\right) \dot t(s)\, ds$$

Switching the vairables we get

$$\int_{t(\alpha)}^{t(\beta)} f(\gamma(t))\dot\gamma(t) dt = \int_\gamma f(z) dz$$

□

Claim 4.

$$\left|\int_\gamma f dz\right| \leq \max_\gamma |f| \cdot L(\gamma)$$

Proof.

$$\left|\int_\gamma f dz\right| = \left|\int_a^b f\left(\gamma(t)\right) \dot\gamma(t)\, dt\right| \leq \int_a^b \left|f\left(\gamma(t)\right)\right| \left|\dot\gamma(t)\right| dt \leq \max_\gamma |f| \int_a^b \left|\dot\gamma(t)\right| dt = L(\gamma)$$

□

Corollary 2. $f_n \xrightarrow{u} f \Rightarrow \int_\gamma^{f_n} \to \int_\gamma f$

Proof.

$$\left|\int_\gamma f_n - \int_\gamma f\right| = \left|\int_\gamma f_n - f\right| \leq \max_\gamma \overbrace{|f_n - f|}^{\to 0} \cdot L(\gamma)$$

□

1.

$$\int_{-\infty}^{\infty} e^{t^2} \cos(2bt)\, dt = \sqrt{\pi} e^{-b^2}$$

2.

$$\int_{-\infty}^{\infty} \cos\left(t^2\right) dt = \int_{-\infty}^{\infty} \sin\left(t^2\right) dt = \frac{\sqrt{\pi}}{2\sqrt{2}}$$

3.

$$\int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$$

4.

$$I = \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$$

5.

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy = \iint_{\mathbb{R}^2} e^{-\left(x^2+y^2\right)} dxdy = \int_0^{\infty} \int_0^{2\pi} e^{-r^2} \cdot rdrd\theta$$

$$= \int_0^{\infty} e^{-r^2} rdr \cdot l = \left[\begin{array}{c} s = r^2 \\ ds = 2rdr \end{array} \right] = \frac{1}{2} \int_0^{\infty} e^{-s} ds \cdot 2\pi = \pi$$