Complex Function Theory

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Theorem 1 (Inverse function)**.** *Let $g \subset \mathbb{C}$ be a domain. $f \in A(G), f : G \xrightarrow[1-1]{onto} G' \Rightarrow f = f^{-1} \in A(G), g(w) = \frac{1}{f'(g(w))}$ $(f'(z) \neq 0)$.*

Proof. Next month ⌣̈ □

1 Series

Let $\{a_n\} \subset \mathbb{C}$, $\sum_n a_n$.

Theorem 2 (Cauchy criterion)**.**

$$\sum_n a_n \text{ converges} \Leftrightarrow \forall \varepsilon > 0. \exists N. \forall m, n \geq N. \left| \sum_{k=n}^{m} a_k \right| < \varepsilon$$

Definition 1 (Absolute convergence)**.** A series $\{a_n\}$ is said to absolutely converge if $\sum_n |a_n|$ converges.

Theorem 3. *If a series absolutely converges, changing the order of summation doesnt change the sum.*

If $A = \sum_n a_n, B = \sum bb_m$ absolutely converge, then $\sum_{m,n} a_n b_m$ absolutely converges to AB

Example 1.

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} \cdot \frac{1}{m!} \qquad \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_{m=0}^{\infty} \frac{w^m}{m!} = \sum_{N=0}^{\infty} \frac{(z+w)^N}{N!}$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \cdot \sum_{m=0}^{\infty} (-1)^m \frac{z^{2m}}{(2m)!} =$$

1.1 Uniform convergence

Let $\star = \sum_n a_n z^n$ where $z \in E \subset \mathbb{C}$, $a_n \mathbb{C} \to \mathbb{R}$. We will assume that $\sum_n m_n < \infty$m $\sup_E |a_n| \leq m_n$ then $\star$ uniformly converges (and absolutely).

Example 2. $\sum_{n=0}^{\infty} z^n$ converges in $|z| \leq const < 1$.
$\sum_{n=0}^{\infty} \left(\frac{z-1}{z+1}\right)^n$ converges when $\left|\frac{z-1}{z+1}\right| < 1$ meaning that it converges uniformly and absolutley in every compact $K \subset \left\{\Re(z) > 0\right\}$

Definition 2. A series $\sum_{n=0}^{\infty} a_n z^n$ converges **normally** in the domain G if it convergews uniformly and absolutely in every compact $K \subset G$.

2 Power series

A series around z_0 looks like $\sum_{n=0}^{\infty} a_n (z-z_0)^n$, $(a_n)_n = 0^{\infty} \subseteq \mathbb{C}, z_0 \in \mathbb{C}$ and $(A) := \sum_{n=0}^{\infty} a_n z^n$ we will work WLOG with $z_0 = 0$.

Theorem 4 (Cauchy hadamard)**.** *The convergence radius is* $\frac{1}{R}\overline{\lim_{n\to\infty}}|a_n|^{1/n}$

Theorem 5. *1. A series (A) converges in $\triangle_R := \{|z| < R\}$.*

2. A series converges in $\{|z| > R\} = \mathbb{C}\backslash\overline{\triangle_R}$

3. $f(z) := \sum_{n=0}^{\infty} a_n z^n$ $f \in A(\triangle_R)$

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n := (A)'$$

Corollary 1. *1. $R_{A'} = R_A$*[1]

2. All of the derivatives $f^{(n)}$ are analytic in $\triangle_R$ and $a_n = \frac{f^{(n)}(0)}{n!}$

Proof. 1. Let $q < 1$ we need to prove that (A) uniformly converges in $\{|z| \le qR\}$. Let $\varepsilon < 0$ be small such that $q(1+\varepsilon) < 1$. Let $z \in \mathbb{C}, |z| \le qR$

$$|a_n z^n| \le |a_n| q^n R^n \le ((1+\varepsilon) q)^n$$

Where the second inequality comes from

$$\forall n \ge n_\varepsilon : |a_n| \le \left(\frac{1-\varepsilon}{R}\right)^n$$

2. If $|z| < R$ then

$$\forall \varepsilon > 0 \exists (n_j) \to \infty \left|a_{n_j}\right| > \left(\frac{1-\varepsilon}{R}\right)^{n_j}$$

WLOG $|z| \ge qR, q > 1$. Then $(1-\varepsilon) q > 1$ for ε which is small enough.

$$\left|a_{n_j} z^{n_j}\right| \ge \left(\frac{1-\varepsilon}{R}\right)^{n_j} (qR)^{n_j} = ((1-\varepsilon) q)^{n_j} \xrightarrow{n\to\infty} \infty$$

3.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, f_1(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

From Calculus 1, $R_{f_1} = R_f$.
Using the taylor expansion:

$$f(z) = \overbrace{\sum_{n=0}^{N} a_n z^n}^{:=p_N} + \overbrace{\sum_{n=N+1}^{\infty} a_n z^n}^{:=r_N}$$

$f_1 = \lim N \to \infty p'_N(z)$

$$\frac{f(z)-f(\xi)}{z-\xi} - f_i(\xi) = \overbrace{\frac{p_N(z)-p_N(\xi)}{z-\xi} - p'_N(\xi)}^{:=I} + \cdots + \overbrace{p'_n(\xi) - f_1(\xi)}^{:=II} + \overbrace{\frac{R_N(z)-R_N(\xi)}{z-\xi}}^{:=III}$$

[1] $\overline{\lim}\left((n+1)|a_{n+1}|\right)^{1/n} = \overline{\lim}|a_n|^{1/n}$

WLOG $|\xi| \leq \rho R$, $z \to \xi$ it is obvious that $I, II \to 0$. As for $III\ldots$

$$III = \left|\frac{R_N(z) - R_N(\xi)}{z - \xi}\right| = \left|\sum_{n>N} a_n \cdot \frac{z^n - \xi^n}{z - \xi}\right| = \left|\sum_{n>N} a_n \left(z^{n-1} + z^{n-2}\xi + \cdots + \xi^{n-1}\right)\right| \leq \cdots \leq \sum_{n>N} |a_n| \, n\rho^{n-1} <$$

Which converges since $\rho < R$.

$\square$